

Screening in strongly coupled $\mathcal{N} = 2^*$ supersymmetric Yang-Mills plasma

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ABSTRACT: Using gauge-gravity duality, we extend thermodynamic studies and present results for thermal screening masses in strongly coupled $\mathcal{N} = 2^*$ supersymmetric Yang-Mills theory. This non-conformal theory is a mass deformation of maximally supersymmetric $\mathcal{N} = 4$ gauge theory. Results are obtained for the entropy density, pressure, specific heat, equation of state, and screening masses, down to previously unexplored low temperatures. The temperature dependence of screening masses in various symmetry channels, which characterize the longest length scales over which thermal fluctuations in the non-Abelian plasma are correlated, is examined and found to be asymptotically linear in the low temperature regime.

KEYWORDS: Gauge-gravity correspondence

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1 Introduction

A fundamental characteristic of any system in thermal equilibrium is the correlation length: the longest distance scale over which spatial fluctuations are significantly correlated. More precisely, the correlation length describes the slowest exponential fall-off of correlation functions at asymptotically large separation. One may choose to restrict attention to correlators of operators with specified symmetries, and define correlation lengths in individual symmetry channels. Inverse correlation lengths have units of energy and are commonly referred to as *screening masses*. These describe the long distance fall-off of correlations in fluctuations with specified quantum numbers, and hence characterize the nature of infrared effective degrees of freedom.

In, for example, a relativistic QED plasma, Debye screening leads to exponential fall-off of the electric field induced by an external test charge, $\langle \vec{E}(\mathbf{x}) \rangle \propto e^{-|\mathbf{x}|/\xi}$, and corresponding behavior in the correlator of electric field fluctuations. The inverse correlation length, or Debye screening mass, $m_D \equiv 1/\xi$, is $O(eT)$. Fermionic fluctuations in a weakly coupled relativistic plasma have correlations which decrease exponentially with distance on a shorter $O(1/T)$ length scale, corresponding to the inverse of the lowest fermionic Matsubara frequency. Static magnetic fields are not screened at long distances, so the associated correlation length is infinite.

In a non-Abelian QCD plasma, one may isolate the physics of color-electric screening in a gauge-invariant and non-perturbative manner by examining correlators of operators which

are odd under Euclidean time reflection [1]. The Debye screening length can be defined as the longest correlation length in time-reflection odd symmetry channels.¹ At asymptotically high temperatures the Debye screening length may be calculated perturbatively. For an $SU(N_c)$ gauge group with N_f Dirac fermions one finds, at leading order, $m_D = [\frac{1}{3}N_c + \frac{1}{6}N_f]^{1/2} g(T)T$. Long distance properties of QCD at high temperature are effectively described by three-dimensional pure Yang-Mills gauge theory with a dimensionful coupling $g_3^2 \equiv g(T)^2 T$. This theory exhibits three-dimensional confinement and the generation of a non-perturbative $O(g_3^2)$ mass gap. Therefore, the longest distance correlations in asymptotically high temperature QCD are associated with static magnetic fluctuations, and have an $O[(g(T)^2 T)^{-1}]$ correlation length.

At non-asymptotic temperatures, reliable calculations of QCD screening masses require numerical lattice simulations.² At temperatures of a few times the deconfinement temperature, $T/T_c \sim 1-4$, which is the temperature range probed by heavy ion collisions, the dynamics of QCD plasma is strongly coupled. Evidence for the strongly coupled nature of QCD plasma in this regime includes the results for screening masses obtained from lattice simulations (see ref. [3] for a review), as well as the success of low viscosity hydrodynamic simulations in reproducing the collective flow observed at RHIC, and the observed suppression of high transverse momentum jets in RHIC collisions [4].

Numerical lattice simulations are limited to equilibrium Euclidean observables; with very limited exceptions, non-equilibrium Minkowski space dynamics cannot be extracted reliably from lattice simulations. Consequently, there has been much interest in studying dynamic properties of non-Abelian gauge theories which mimic some aspects of QCD and to which the techniques of gauge/gravity (or AdS/CFT) duality [5–7] may be applied. (See, for example, refs. [8–16].) Gauge/gravity duality reformulates the strong coupling, large N_c dynamics of a suitable gauge theory in terms of classical supergravity in an asymptotically anti-de Sitter spacetime. The simplest, and best understood, example is maximally supersymmetric $SU(N_c)$ Yang-Mills theory ($\mathcal{N} = 4$ SYM). Although this theory contains adjoint representation fermions and scalars not present in QCD, at non-zero temperature the resulting non-Abelian plasma shares many qualitative similarities with the quark-gluon plasma of QCD. Screening masses obtained via gauge-gravity duality for strongly coupled ($\lambda \equiv g^2 N_c \gg 1$) $\mathcal{N} = 4$ SYM, in the $N_c \rightarrow \infty$ limit, were discussed in refs. [17, 18]. The conformal invariance of $\mathcal{N} = 4$ SYM implies that thermal screening masses are exactly proportional to T . Ref. [17] compared the values of screening masses of strongly coupled $\mathcal{N} = 4$ SYM with results of lattice QCD simulations at $T \approx 2T_c$. In QCD, this temperature lies within the window $1.5T_c \leq T \lesssim 4T_c$ where screening masses are well described as scaling linearly with T [19, 20]. Ratios of screening masses in different symmetry channels compare rather well between QCD and $\mathcal{N} = 4$

¹Euclidean time reflection R_τ corresponds to the product \mathcal{TC} of time reversal and charge conjugation. This definition of the Debye screening length applies only to \mathcal{TC} -invariant equilibrium states in \mathcal{TC} -invariant theories.

²For example, up to temperatures as high as $10^7 T_c$, the non-perturbative $O(g^2 T)$ correction to the Debye screening mass in QCD is larger than the leading $O(gT)$ perturbative contribution [2].

SYM but, in absolute size, the $\mathcal{N}=4$ SYM screening masses (divided by T) are roughly a factor of two larger than in QCD [17].

It is not currently known whether quantitative differences between QCD and $\mathcal{N}=4$ SYM plasma properties are dominantly due to (i) the differing constituents of the non-Abelian plasmas, (ii) the comparison of QCD at $T/T_c \approx 1.5\text{--}4$, where it is neither weakly coupled nor infinitely strongly coupled, with the $\lambda \gg 1$ limit of SYM instead of some $O(1)$ value of λ , or (iii) the comparison with $N_c = \infty$ SYM instead of $N_c = 3$. Available lattice results suggest that the N_c dependence of plasma thermodynamics and screening masses is quite mild [19, 21], so possibility (iii) is unlikely to be dominant. Possibility (ii) could be studied by computing $1/\lambda$ corrections to the leading strong coupling behavior described by classical supergravity. However, for many observables this requires knowledge of higher derivative corrections to the supergravity action which have not yet been fully worked out. The possibility most accessible to study is (i): the dependence of various observables on the particular constituents of a non-Abelian plasma. Part of the reason that screening masses are larger in $\mathcal{N}=4$ SYM than in QCD is certainly the fact that there are more active degrees of freedom contributing to screening in an $\mathcal{N}=4$ SYM plasma — adjoint representation fermions and scalars instead of fundamental representation quarks. One would expect that reducing the number of light degrees of freedom would yield screening masses which are closer to those of a QCD plasma. This may be explored, in strongly coupled plasmas, by studying theories with known gravity duals which are more QCD-like than $\mathcal{N}=4$ SYM. Examining non-conformal deformations of $\mathcal{N}=4$ SYM will shed light on the sensitivity of ratios of screening masses to departures from conformal invariance.

In this paper, we explore these issues by computing screening masses and thermodynamics of a mass-deformation of maximally supersymmetric Yang-Mills theory known as $\mathcal{N}=2^*$ SYM [22–24]. This theory differs from $\mathcal{N}=4$ SYM by the addition of mass terms and interactions for an adjoint representation hypermultiplet, in a manner which preserves $\mathcal{N}=2$ supersymmetry. This theory has one new free parameter, the deformation mass m . The theory is non-conformal, with a nontrivial renormalization group fixed point (with adjustable coupling λ) at asymptotically high energies. An intrinsic strong scale Λ appears, analogous to Λ_{QCD} , but for large λ this scale is comparable to the mass deformation, $\Lambda \sim m$ [22].

The study of $\mathcal{N}=2^*$ gauge theory at non-zero temperature using holographic methods was pioneered by Buchel, Liu, and collaborators [24–26]. Various properties of $\mathcal{N}=2^*$ plasma have been studied, including shear and bulk viscosities [25, 27, 28], as well as heavy quark drag, momentum broadening, and jet quenching [29]. Reference [26] laid the groundwork for numerically constructing the supergravity background dual to the thermal equilibrium state of $\mathcal{N}=2^*$ SYM, and mapped out the dictionary that translates between supergravity fields and observables in the quantum field theory. Of direct relevance to our work is their result for the free energy density in the range $0 \leq m/T \leq 6$.³ Noticeable deviation from the behavior of $\mathcal{N}=4$ plasma was estimated to occur near the upper end of this range.

³In the later work [27], hydrodynamic coefficients were evaluated up to $m/T \simeq 12$.

We extend the results of ref. [26] on $\mathcal{N}=2^*$ thermodynamics to substantially lower temperatures, $m/T \lesssim 33$, and provide evidence that this data allows one to probe the asymptotically low temperature regime. We argue that, despite the presence of a non-zero mass scale m , this theory (in the large N_c and large λ limit) has no thermal phase transition at any non-zero temperature.

Our primary goal is the evaluation of thermal screening masses for multiple symmetry channels throughout the temperature range $0 \leq m/T \lesssim 33$. We extrapolate the screening mass curves to vanishingly small temperatures, $T/m \rightarrow 0$, where components of the heavy hypermultiplet decouple from the dynamics due to exponential Boltzmann suppression, and extract results for the limiting plasma which contains only light degrees of freedom.

The paper is structured as follows. In Sec. 2, we briefly review the field theoretic formulation of $\mathcal{N}=2^*$ gauge theory and its gravitational representation. In Sec. 3, we discuss equilibrium thermodynamic quantities such as the entropy density, pressure, specific heat, and equation of state. We compute screening masses from linearized fluctuations of supergravity modes in Sec. 4. In Sec. 5, we compare the screening masses of $\mathcal{N}=2^*$ SYM to those of $\mathcal{N}=4$ SYM and to available lattice QCD data. The holographic renormalization and derivation of thermodynamic quantities from the gravitational dual are discussed, at some length, in Appendix A. The numerical method we used to find the background geometry is explained in Appendix B, and fluctuations of the gravitational fields we use to compute screening masses are described in Appendix C. Additional material related to the appendices can be found online [30].

2 $\mathcal{N}=2^*$ gauge theory and its gravity dual

$\mathcal{N}=2^*$ supersymmetric Yang-Mills theory is a mass deformation of $\mathcal{N}=4$ SYM. We consider the theory on $\mathbb{R}^{1,3}$ with gauge group $SU(N_c)$ in the 't Hooft limit: $N_c \rightarrow \infty$ and $g^2 \rightarrow 0$ with $\lambda \equiv g^2 N_c$ held fixed. If the $\mathcal{N}=4$ field content is grouped into $\mathcal{N}=1$ superfields, there is a vector multiplet and three adjoint chiral multiplets Φ_i ($i=1,2,3$). The Lagrange density of $\mathcal{N}=2^*$ SYM is obtained by adding the term

$$\delta\mathcal{L} = m \int d^2\theta \operatorname{tr}(\Phi_1\Phi_2) + \text{h.c.} \quad (2.1)$$

to the $\mathcal{N}=4$ SYM Lagrange density $\mathcal{L}_{\mathcal{N}=4}$. Appropriate field redefinitions can remove any phase in the mass, so m may be taken real and positive without loss of generality. The two chiral fields (Φ_1, Φ_2) comprise a single $\mathcal{N}=2$ massive hypermultiplet, and the remaining fields form an $\mathcal{N}=2$ vector multiplet. Once auxiliary field constraints have been solved, the above contribution to the superpotential induces conventional mass terms for two Weyl fermions and two complex scalars, as well as terms trilinear in the scalars. In order to identify the gravity dual of the $\mathcal{N}=2^*$ SYM theory, it is convenient to write the mass deformation in terms of relevant operators in irreducible representations of the $\mathcal{N}=4$ R -symmetry group $SO(6)_R$. We can distinguish two contributions, first there is an irreducible scalar mass

$$\mathcal{O}_2 \equiv \frac{1}{3} \operatorname{tr}(-|\phi_1|^2 - |\phi_2|^2 + 2|\phi_3|^2). \quad (2.2)$$

This operator, which is an element of the $\mathbf{20}'$ of $SO(6)$, appears in the 4D (Euclidean) $\mathcal{N}=2^*$ action $S_{\mathcal{N}=2^*}$ as $-(2/g^2) \int d^4x m^2 \mathcal{O}_2$. Hence, it contributes the usual positive bosonic mass terms for the hypermultiplet scalars ϕ_1 and ϕ_2 . But it also destabilizes the scalar ϕ_3 belonging to the vector multiplet. The second contribution introduces a mass for the fermions in the hypermultiplet

$$\mathcal{O}_3 \equiv \text{tr} (i\psi_1\psi_2 - \sqrt{2}\phi_3[\phi_1, \phi_1^\dagger] - \sqrt{2}\phi_3[\phi_2^\dagger, \phi_2] + \text{h.c.}) - \frac{2}{3} m \text{tr} (|\phi_1|^2 + |\phi_2|^2 + |\phi_3|^2), \quad (2.3)$$

and appears in $S_{\mathcal{N}=2^*}$ as $-(2/g^2) \int d^4x m \mathcal{O}_3$. One may obtain the expression for \mathcal{O}_3 by first deriving the on-shell Lagrange density for mass-deformed $\mathcal{N}=4$ SYM in terms of component fields, isolating the m -dependent terms, and subtracting $m^2 \mathcal{O}_2$ from them. Note that \mathcal{O}_3 is modified, from its massless limit, by the $SO(6)$ singlet operator $\sum_1^3 |\phi_i|^2$ multiplied by a factor of m . This term is crucial since it cancels the negative potential energy for ϕ_3 introduced by \mathcal{O}_2 . It is required by the central extension of the $\mathcal{N}=2$ supersymmetry algebra.⁴

Since the UV fixed point describes the conformal $\mathcal{N}=4$ theory, whose coupling is a free parameter, one has the freedom to choose the UV coupling g^2 (or λ) arbitrarily.⁵ We choose the 't Hooft coupling to be large, $\lambda \gg 1$, but finite as $N_c \rightarrow \infty$.⁶ The value of this coupling, together with the mass m , determines the intrinsic scale where the theory becomes strongly coupled, $\Lambda \sim m e^{-4\pi^2/\lambda}$ [22]. Notice, however, that for strong coupling the scale Λ is comparable to m and there is no large separation of scales. Consequently, one cannot decouple massive modes at the scale m while preserving the low energy spectrum at the scale Λ .

The global symmetry group is $SU(2)_R \times U(1)$. The $U(1)$ is an ordinary flavor symmetry under which Φ_1 has charge $+1$, Φ_2 has charge -1 , and Φ_3 is neutral. The superpotential term $m \text{tr} (\Phi_1 \Phi_2)$ does not preserve the classical $U(1)_R$ symmetry that exists in the massless $\mathcal{N}=4$ theory. However, one can regard m as the expectation value of a background field and assign

⁴To see this, consider the identity $\{Q_\alpha^A, Q_\beta^B\} = \epsilon_{\alpha\beta} Z^{AB}$. To obtain a massive multiplet which has 4 helicity states (2 integer and 2 odd half-integer), the Bogomolny bound must be saturated: $Z^{12} = -Z^{21} = 2m$. This value for the antisymmetric matrix Z^{AB} forces half of the $2\mathcal{N}$ fermionic “oscillators” (obtained from suitable linear combinations of the supercharges) to obey trivial anticommutation relations. This means that they cannot be used to create or annihilate additional helicity states. This, in turn, shortens a massive irreducible representation of the \mathcal{N} -extended super-Poincaré algebra from dimension $2^{2\mathcal{N}}$ to $2^\mathcal{N}$, which is precisely what is needed to have a massive hypermultiplet. The action of Q^1 and Q^2 applied successively to \mathcal{O}_2 must yield the bosonic operator \mathcal{O}_3 , and the anticommutation relation implies that an explicit factor of m must appear in \mathcal{O}_3 .

The dual operator analogous to \mathcal{O}_3 in the case of fundamental representation hypermultiplets is given explicitly in Appendix A of ref. [31]. Specifically, in eq. (A.1) of this reference, the operator \mathcal{O}_m has an identical fermion bilinear, similar scalar trilinears, and similar mass terms for the hypermultiplet scalars. However, unlike \mathcal{O}_3 , commutators do not appear in the scalar trilinears of \mathcal{O}_m . The reason is that their superpotential (in our notation) is $\text{tr} (\Phi_1 \Phi_3 \Phi_2)$ rather than $\text{tr} (\Phi_3 [\Phi_1, \Phi_2])$. We thank Andreas Karch for pointing this out to us.

⁵One may also choose an arbitrary θ angle, but this will not be important in our discussion.

⁶This is necessary since the supergravity approximation to the string partition function is only valid for spacetime curvatures which are small when measured in string units. In our case, the ratio ℓ_s/L of the fundamental string length ℓ_s to the characteristic curvature radius L equals $\lambda^{-1/4}$.

it an R -charge of 2. This implies that the low energy effective action must be holomorphic in m . The vacuum manifold is an (N_c-1) -dimensional complex space specified by adjoint scalar expectation values having the form $\langle \Phi_1 \rangle = \langle \Phi_2 \rangle = 0$ and $\langle \Phi_3 \rangle = \text{diag}(a_1, \dots, a_{N_c})$ (up to a gauge transformation), subject to the tracelessness constraint, $\sum_i a_i = 0$. At a generic point in moduli space, $\mathcal{N}=2$ supersymmetry is unbroken, the gauge group is Higgsed to $U(1)^{N_c-1}$, and the phase is Coulombic. A Wilsonian effective action for the low-energy Abelian theory may be generated by integrating out massive W bosons, their superpartners, and the adjoint hypermultiplet. Two derivative terms in the effective action are determined from a prepotential which depends holomorphically on the complexified gauge coupling constant $\tau \equiv \theta/(2\pi) + 4\pi i/g^2$, the mass m , and the eigenvalues $\{a_i\}$ [32]. The prepotential receives classical and quantum contributions. The perturbative correction is one-loop exact and may be determined by a matching calculation. Nonperturbative corrections become important in regions of moduli space where BPS states become light. Both monopole and W boson masses are proportional to differences of scalar eigenvalues, $|a_i - a_j|$, and vanish when eigenvalues coincide. Away from points of eigenvalue degeneracy, nonperturbative corrections are suppressed and the low energy dynamics is well-described by the perturbative prepotential.

In gauge/gravity duality, certain gravitational backgrounds may be interpreted as dual descriptions of $\mathcal{N}=4$ SYM perturbed by relevant operators. In particular, solutions of 5D maximally supersymmetric $\mathcal{N}=8$ gauged supergravity which asymptotically approach AdS_5 are dual to states of $\mathcal{N}=4$ SYM with gauge group $SU(N_c)$. If fields in addition to the metric are non-vanishing, then one can obtain non-conformal boundary theories with non-trivial renormalization group flow. Because 5D gauged supergravity is a consistent truncation of 10D type IIB supergravity on $AdS_5 \times S^5$, it is sufficient to solve the dimensionally-reduced problem (as opposed to the full 10D problem) [33–35]. It is possible to unambiguously “uplift” any 5D solution to a full 10D solution although, in practice, finding the uplift is nontrivial. The scalar and gravity part of the 5D action has the form

$$S_{5D} = \frac{1}{4\pi G_5} \int d^5x \sqrt{-g} \left(\frac{1}{4} R + \mathcal{L}_{\text{matter}} \right). \quad (2.4)$$

Specific terms in $\mathcal{L}_{\text{matter}}$ will be discussed below. Massless bosonic modes in 10D may be expanded in a set of Kaluza-Klein modes on the S^5 , with masses of order $1/L$, where L is the radius of the five-sphere. There are 42 scalar fields and a complicated potential V which depends on 40 of them [33–35]. For a thorough description of the subsector we will study here, see refs. [23, 36]. The scalars fall into various representations of the $SO(6)$ gauge group: there is a $\mathbf{20}'$ representation with mass-squared $M^2 = -4/L^2$, a $\mathbf{10} \oplus \overline{\mathbf{10}}$ representation with $M^2 = -3/L^2$, and dilaton and axion singlets which have $M^2 = 0$ and do not contribute any potential energy. The scaling dimension Δ of a CFT operator \mathcal{O}_Δ dual to a supergravity scalar field with mass-squared M^2 on AdS_5 is given by $\Delta(\Delta - 4) = M^2 L^2$. The $SO(6)$ symmetry maps to the global R -symmetry in the $\mathcal{N}=4$ SYM theory. The $\mathbf{20}'$ is dual to dimension 2 symmetric traceless combinations of the six $\mathcal{N}=4$ scalars (i.e., the real and imaginary parts of ϕ_i), the $\mathbf{10} \oplus \overline{\mathbf{10}}$ is dual to dimension 3 symmetric bilinears of the four

$\mathcal{N}=4$ Weyl fermions plus supersymmetric completions, and the dilaton-axion pair is dual to the $\mathcal{N}=4$ Lagrange density plus theta term.⁷ The scalar fields have $M^2 \leq 0$ (but above the Breitenlohner-Freedman stability bound), which corresponds to $\Delta \leq 4$. Therefore, a solution of 5D $\mathcal{N}=8$ gauged supergravity with non-vanishing profiles for any of the 40 nontrivial scalars corresponds to a relevant deformation of the boundary CFT.

In our case, the deformations are encoded in two real supergravity scalars α and χ , in terms of which the 5D matter Lagrange density equals

$$-\mathcal{L}_{\text{matter}} = 3(\partial\alpha)^2 + (\partial\chi)^2 + V(\alpha, \chi), \quad (2.5)$$

with

$$V(\alpha, \chi) = \hat{g}^2 \left[-\frac{1}{4}e^{-4\alpha} - \frac{1}{2}e^{2\alpha} \cosh 2\chi + \frac{1}{16}e^{8\alpha} \sinh^2 2\chi \right]. \quad (2.6)$$

The dimensionful gauged supergravity coupling is $\hat{g}^2 = (2/L)^2$, and the 5D Newton's constant is $G_5 = \pi L^3/(2N_c^2)$. The field α has $M^2 = -4/L^2$ and is dual to the $\mathcal{N}=2^*$ operator \mathcal{O}_2 in eq. (2.2). The second scalar field χ has $M^2 = -3/L^2$ and is dual to the $\mathcal{N}=2^*$ operator \mathcal{O}_3 in eq. (2.3). The value of the mass m in the field theory is determined by the asymptotic boundary conditions of the fields α and χ , which are not independent but related by supersymmetry. Notice that the identification of χ with \mathcal{O}_3 implies that the map between the field and the dual operator depends on the choice of boundary conditions/couplings in the field theory Lagrangian. This modification of the usual prescription may be a consequence of using a Kaluza-Klein compactification of ten-dimensional supergravity, we consider preserving supersymmetry on both sides as an indication that the identification is correct.

A solution to 5D $\mathcal{N}=8$ gauged supergravity with non-vanishing scalars α and χ is known, and its full 10D uplift has been constructed [23]. The bulk 5D geometry has four-dimensional Poincaré invariance. An analysis using a slowly-moving probe D3-brane indicates that the background branes form a locus around the origin of the probe's moduli space (the space where the probe's potential vanishes) where the probe's kinetic energy vanishes. This ‘*enhançon*’ geometry is dual to a special vacuum of the $\mathcal{N}=2^*$ Coulomb branch where extra dyonic states become massless [32, 37]. We will see that this solution plays an important role in the low temperature regime.

3 Equilibrium thermodynamics

Buchel, Deakin, Kerner, and Liu numerically solved the 5D $\mathcal{N}=8$ gauged supergravity equations for a background with three-dimensional rotational invariance and a regular black brane horizon [26].⁸ They also verified that the 10D uplift solves the IIB supergravity equations of motion. This geometry is dual to a thermal equilibrium state of strongly coupled $\mathcal{N}=2^*$

⁷The boundary value of the 5D dilaton-axion pair is the UV marginal coupling τ . The physical running coupling, dual to the 10D dilaton and axion, is a nontrivial function of τ , m , and the energy scale [23].

⁸The zero temperature solutions were previously constructed analytically by Pilch and Warner [23].

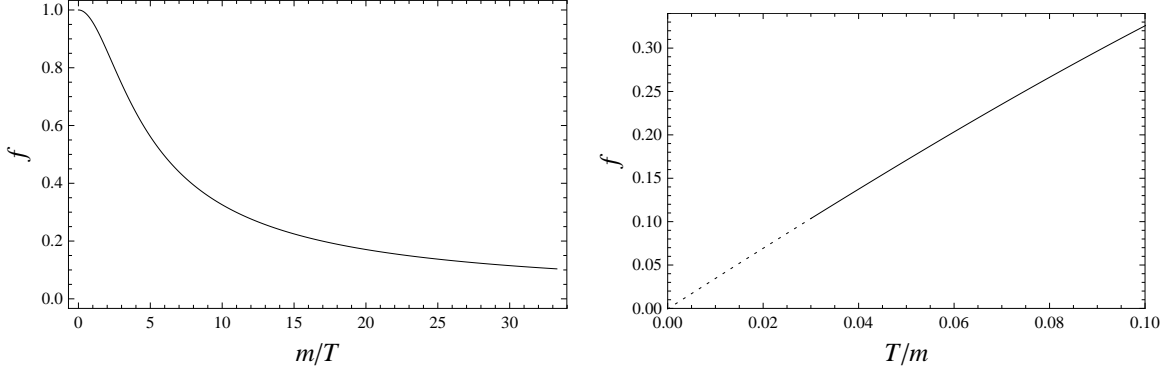


Figure 1. Left: Pressure divided by $\frac{\pi^2}{8} N_c^2 T^4$, as a function of m/T . Right: Low temperature behavior plotted as a function of T/m . The dotted line shows the extrapolation of a quadratic fit to the ten lowest-temperature data points.

SYM with gauge group $SU(N_c)$ in the large N_c limit. There is a single dimensionless parameter, m/T , which may be freely adjusted. Thermodynamic quantities, when divided by appropriate powers of T , must be dimensionless functions of m/T .

We have extended the analysis of ref. [26] to a wider range of m/T . The holographic representation of the entropy and free energy is summarized in Appendix A. As m/T increases, the differential equations one must solve become increasingly stiff, making the numerical exploration of very large values of m/T challenging.

The pressure (equal to minus the free energy density) may be expressed in the form

$$p = \frac{\pi^2}{8} N_c^2 T^4 f(m/T). \quad (3.1)$$

The scaling function f is plotted in figure 1. The free energy is obtained from the renormalized Euclidean supergravity action (see Appendix A). The prefactors in eq. (3.1) have been chosen to equal the $\mathcal{N}=4$ value of the pressure, so that $f(0) = 1$. A high temperature expansion yields $f = 1 - 2\pi^{-4} \Gamma(\frac{3}{4})^4 (m/T)^2 + O[(m/T)^3]$ [24]. Our numerical computations extend out to $m/T \approx 33.3$, at which point $f \approx 0.1037$. Throughout the range of temperatures $0 \leq m/T \lesssim 33$, our numerically-determined black brane solution describes a thermodynamic state which is at least locally stable. Both the pressure and the specific heat (discussed below) are positive, and we presume that our solution is describing the true equilibrium state. As shown in the right-hand plot of figure 1, extrapolation of our numerical results clearly suggest that f vanishes linearly as $T/m \rightarrow 0$, implying that the free energy density $F/\mathcal{V} = O(T^5/m)$ at low temperature. The interpretation of this scaling behavior is discussed below.

The entropy density may be expressed as

$$S/\mathcal{V} = \frac{\pi^2}{2} N_c^2 T^3 \sigma(m/T), \quad (3.2)$$

where the prefactor equals the entropy density of $\mathcal{N}=4$ SYM [38], so that $\sigma(0) = 1$. The entropy may be obtained from the horizon area (A.8) or from a thermodynamic derivative

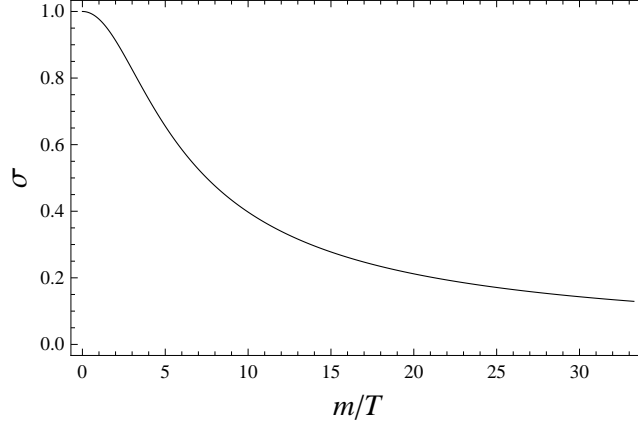


Figure 2. The entropy density, divided by $\frac{\pi^2}{2} N_c^2 T^3$, as a function of m/T .

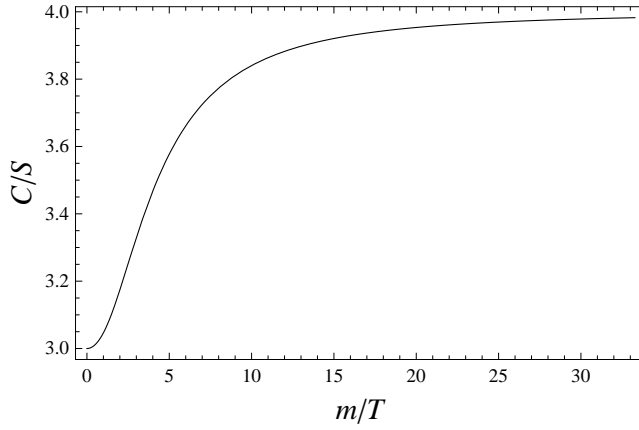


Figure 3. The specific heat to entropy ratio, $C_V/S = \partial(\ln S)/\partial(\ln T)$, as a function of m/T .

of the free energy, $S = -\partial F/\partial T$, which implies that $\sigma = f - \frac{1}{4} \frac{m}{T} f'$. The resulting scaling function σ is plotted in figure 2. For $m/T \ll 1$, a high temperature expansion yields $\sigma = 1 - \pi^{-4} \Gamma(\frac{3}{4})^4 (m/T)^2 + O[(m/T)^3]$ [24]. As the temperature drops, σ decreases and appears to approach zero linearly in T/m , implying that the entropy density $S/\mathcal{V} = O(T^4/m)$ at low temperature.

The specific heat at constant volume is given by

$$C_V = T \left(\frac{\partial S}{\partial T} \right)_V = S \left(3 - \frac{m}{T} \frac{\sigma'}{\sigma} \right). \quad (3.3)$$

A plot of the specific heat to entropy ratio, C_V/S , is shown in figure 3, from which it is evident that the specific heat is always positive. The speed of sound (squared) in the plasma is inversely related to the specific heat, $c_{\text{sound}}^2 = C_V/S$. Figure 3 shows that c_{sound}^2 equals $1/3$ in the high temperature (or massless) limit, as required from conformal invariance, with corrections quadratic in m . Explicitly, $c_{\text{sound}}^2 = \frac{1}{3} - \frac{2}{9} \pi^{-4} \Gamma(\frac{3}{4})^4 (m/T)^2 + O[(m/T)^3]$. At low temperatures, the speed of sound (squared) evidently approaches $1/4$.

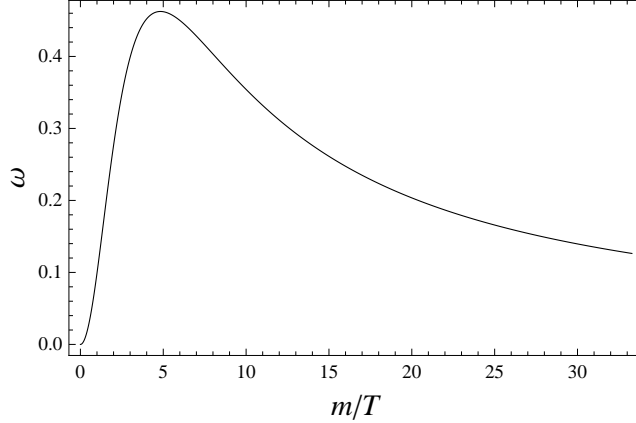


Figure 4. The scaling function for the equation of state, $\omega = (\epsilon - 3p)/(N_c^2 T^4)$, as a function of m/T .

The equation of state, relating the energy density ϵ and pressure p to the temperature, may be expressed as

$$\epsilon - 3p = N_c^2 T^4 \omega(m/T), \quad (3.4)$$

where

$$\omega = \frac{\pi^2}{2} (\sigma - f) = -\frac{\pi^2}{8} \frac{m}{T} f'. \quad (3.5)$$

This function is plotted in figure 4. Since $\epsilon - 3p$ is the trace of the energy-momentum tensor, it vanishes in the conformal limit. For small m/T the scaling function ω rises quadratically, $\omega = \frac{1}{2} \pi^{-2} \Gamma(\frac{3}{4})^4 (m/T)^2 + O[(m/T)^3]$ [24]. As shown in figure 4, ω peaks at $m/T \approx 4.83$ and decreases monotonically thereafter. The position of the peak corresponds to the temperature below which the plasma is cool enough that massive components of the hypermultiplet have a low probability of being thermally excited. The high-mass/low-temperature tail of ω must be predominately due to the non-renormalizable (i.e., higher-derivative) interactions of the light degrees of freedom. The fact that $m/T \approx 33$ lies well out in the tail of ω , far from the peak, is evidence that our numerical results are probing behavior deep into the low temperature regime.

The low temperature regime is dominated by the special enhançon solution mentioned in Sec. 2. This is shown by the plot in figure 5, which displays the evolution of scalars fields $(\chi, e^{6\alpha})$ along the radial direction for solutions at different values of m/T , superimposed on the $T = 0$ enhançon solution, whose explicit form can be found in Appendix A. As the temperature drops to zero, our solutions clearly approach the enhançon solution. As discussed in ref. [39], the dual to the enhançon geometry may be interpreted as a flow from the $\mathcal{N} = 4$ SYM theory in the UV to a five-dimensional CFT in the IR. At low temperatures the effective five-dimensional theory dominates the dynamics, so this explains the behavior of the free energy, $F/\mathcal{V} \sim T^5/m$, the entropy, $S/\mathcal{V} \sim T^4/m$, and the speed of sound, $c_{\text{sound}}^2 \rightarrow 1/4$, that we observe.⁹

⁹This is also consistent with the result of refs. [27, 28] that in the zero temperature limit the bulk viscosity ζ saturates the bound $\zeta/\eta \geq 2(\frac{1}{3} - c_s^2)$, in agreement with ref. [40].

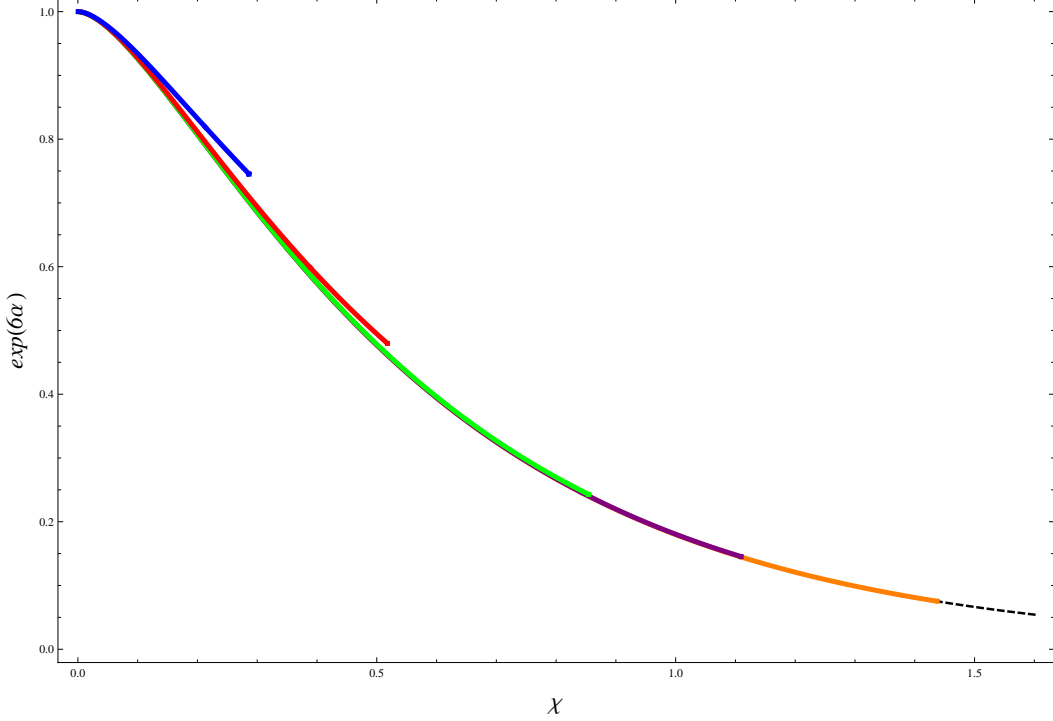


Figure 5. Evolution of the scalar fields $(\chi, e^{6\alpha})$ as the radial coordinate varies. All curves start at $(0, 1)$, corresponding to the boundary where the scalars α and χ vanish. Curves end at the point $(\chi_h, e^{6\alpha_h})$ corresponding to the value of the scalars at the horizon. The $T = 0$ enhançon solution is shown as a dashed curve extending from $(0, 1)$ to $(\infty, 0)$. Finite temperature solutions are shown as solid lines for $m/T \approx 2.23$ (blue), 4.57 (red), 10.0 (green), 17.1 (purple), and 33.3 (orange). Our finite temperature solutions clearly approach the enhançon solution as $m/T \rightarrow \infty$.

This low-temperature behavior is presumably specific to the $N_c \rightarrow \infty$ limit. Turning on an infinitesimal, non-zero temperature will deform the moduli space of degenerate vacuum states of $\mathcal{N} = 2^*$ theory into some non-trivial free-energy surface. One expects the free energy to be minimized at those points in moduli space with the maximal number of massless degrees of freedom. In $\mathcal{N} = 4$ SYM, a non-zero temperature destabilizes all points on moduli space other than the origin, where all scalar expectation values vanish and the entire $SU(N_c)$ gauge group is “unbroken.” Conformal invariance guarantees that the non-Abelian plasma phase, with $O(N_c^2)$ free energy, extends all the way down to $T = 0$. For the non-conformal $\mathcal{N} = 2$ SYM theory, the low temperature behavior is more subtle. There is no point on the $\mathcal{N} = 2$ moduli space where a non-Abelian gauge field is present in the low energy dynamics. Low energy degrees of freedom consist of $U(1)^{N_c-1}$ gauge fields plus, at special points in moduli space, massless BPS monopoles or dyons [41–43]. One would expect low temperature equilibrium states to correspond to points where a maximal number of BPS dyons simultaneously become massless. This suggests that for sufficiently low temperatures there will be multiple degenerate

equilibrium states, related by $SL(2, \mathbb{Z})$ electro-magnetic duality [42].¹⁰ Hence, for finite N_c , we do expect a distinct low temperature phase to exist for some non-zero range of temperature. However, it is quite plausible that the transition temperature scales as Λ/N_c^α for some $\alpha > 0$. This would be consistent with our numerical results shown in figure 1, and is also consistent with the expected domain of validity of the low-energy description near the degenerating points studied in ref. [44]. Further study of the low temperature behavior of this theory at large but finite values of N_c would be interesting.

4 Screening masses

Using holography, one may compute the thermal screening mass associated with a gauge invariant operator \mathcal{O} by solving for the lowest eigenvalue of the linear operator which describes fluctuations in the corresponding dual field [46–49]. Let us remind the reader of the underlying logic. The long-distance behavior of the Euclidean two-point function of \mathcal{O} determines the screening mass. If the Fourier transform of the correlator, viewed as a function of the spatial wavevector, is analytic in a strip of width κ above the real axis, then the coordinate space correlator will fall with spatial separation $|\mathbf{x}-\mathbf{x}'|$ at least as fast as $e^{-\kappa|\mathbf{x}-\mathbf{x}'|}$. Hence, the screening mass κ equals the distance from the real axis to the nearest singularity. If the Fourier transformed correlator is analytic except for poles on the imaginary axis, as will be the case in the large N_c theory under consideration, then the associated screening mass equals the magnitude of the pole closest to the origin.

There is a well-defined prescription for calculating Green’s functions using gauge/gravity duality. One must find solutions to the *linearized* equations of motion for the fluctuations of the supergravity field dual to \mathcal{O} . We will use combinations of fluctuation fields which are invariant under linearized diffeomorphisms. After a 4D Fourier transform, fluctuation fields depend on the radial coordinate r , the spatial wavevector \mathbf{k} , and a discrete Matsubara frequency $\omega_n \equiv 2\pi nT$. We will limit attention to zero frequency fluctuations, since non-zero frequency fluctuations have shorter correlation lengths than the corresponding static fluctuations.

The resulting linearized fluctuation equations of motion are second order ordinary differential equations (ODEs) in r . For some channels there is a single ODE involving just one fluctuation field; for other channels there is a coupled set of ODEs involving multiple fields. The ODE coefficients depend on the radial profiles of the scalar fields α and χ plus the 5D metric components describing the background geometry, as well as the fluctuation wavevector \mathbf{k} which appears in the dimensionless combination $(\mathbf{k}L)^2$.

For the sake of discussion, consider a gauge invariant supergravity field $Z(r)$ that obeys a second order ODE. A general solution may be expressed as a linear combination of two independent solutions having simple behavior near the horizon, $Z(r) = c_I^{\text{hor}} Z_I^{\text{hor}}(r) + c_{II}^{\text{hor}} Z_{II}^{\text{hor}}(r)$,

¹⁰For $\mathcal{N}=2$ gauge theory, a detailed analysis of the temperature-induced deformation of moduli space has only been done for the case of an $SU(2)$ gauge group [45], where it was argued that the discrete \mathbb{Z}_2 R -symmetry is spontaneously broken at low temperature.

with coefficients c_j^{hor} which will depend on the wavevector \mathbf{k} . The horizon is a regular singular point of the fluctuation ODE and one solution, which we will choose to call $Z_{\text{II}}^{\text{hor}}$, will have a logarithmic singularity at the horizon. The physical solutions in which we are interested should be regular at the horizon, and hence we must set $c_{\text{II}}^{\text{hor}} = 0$. The general solution may also be expressed as a linear combination of two solutions having simple behavior near the boundary ($r \rightarrow \infty$), $Z(r) = c_{\text{I}}^{\text{bdy}} Z_{\text{I}}^{\text{bdy}}(r) + c_{\text{II}}^{\text{bdy}} Z_{\text{II}}^{\text{bdy}}(r)$. We may choose independent solutions such that $Z_{\text{I}}^{\text{bdy}}$ vanishes as $r \rightarrow \infty$ while $Z_{\text{II}}^{\text{bdy}} \rightarrow 1$ as $r \rightarrow \infty$. Hence, the coefficient $c_{\text{II}}^{\text{bdy}}$ equals the asymptotic value of the general solution Z .

The asymptotic value $c_{\text{II}}^{\text{bdy}}$ functions as the source for the (Fourier-transformed) gauge theory operator \mathcal{O} , in accordance with the standard AdS/CFT interpretation. Hence, the Euclidean two-point function for \mathcal{O} is the second functional derivative of the regulated on-shell supergravity action with respect to the value of Z at the $r = \infty$ boundary. The part of the on-shell action quadratic in fluctuations reduces to a boundary term of the form $\lim_{r \rightarrow \infty} \int d^3k \mathcal{F}(\mathbf{k}) Z'(r) Z(r)$ plus contact terms which do not contain $Z'(r)$, where the function $\mathcal{F}(\mathbf{k})$ depends on details of the action [50]. This leads to the result $\langle \mathcal{O} \mathcal{O} \rangle \sim c_{\text{I}}^{\text{bdy}} / c_{\text{II}}^{\text{bdy}}$, up to terms analytic in \mathbf{k}^2 which can be removed by local counterterms.¹¹ Poles of the Green's function correspond to zeros of $c_{\text{II}}^{\text{bdy}}$. Consequently, to determine the screening mass one must find the least negative value of $(\mathbf{k}L)^2$ for which the linearized equation of motion for the fluctuation Z has a solution satisfying a Dirichlet condition at the boundary together with regularity at the horizon, $c_{\text{II}}^{\text{hor}} = c_{\text{II}}^{\text{bdy}} = 0$.

We apply this procedure to evaluate the lightest screening masses for the energy-momentum tensor $T_{\mu\nu}$ of $\mathcal{N}=2^*$ SYM. This operator is dual to fluctuations of the 5D metric in $\mathcal{N}=8$ gauged supergravity. We also obtain screening masses for the $\mathcal{N}=4$ Lagrange density, $\mathcal{L}_{\mathcal{N}=4} = \frac{1}{4} \text{tr} F^{\mu\nu} F_{\mu\nu} + \dots$, dual to the 5D dilaton, as well as the Pontryagin density, $\text{tr} E \cdot B = \frac{1}{4} \text{tr} F^{\mu\nu} \tilde{F}_{\mu\nu}$, which is dual to the 5D axion.

In each Euclidean correlator the operators are spatially separated along the x^3 direction, so field theory coordinates transverse to this longitudinal direction are x^1, x^2 , and τ . Distinct symmetry channels are classified according to irreducible representations of the $O(2)$ group of rotations in the 1-2 plane, together with discrete eigenvalues for Euclidean time reversal (taking $\tau \rightarrow -\tau$), which we will denote as \mathcal{R}_τ , and charge conjugation \mathcal{C} . (The eigenvalues of \mathcal{R}_τ and \mathcal{C} will be denoted by R_τ and C , respectively.) Irreducible representations of $O(2)$ are labeled by a non-negative integer angular momentum (or helicity) \mathcal{J} plus, when $\mathcal{J} = 0$, an eigenvalue for reflections in the 1-2 plane. We will use R_y to denote the eigenvalue of the reflection \mathcal{R}_y taking $x^2 \rightarrow -x^2$. The resulting $\mathcal{J}_{R_y}^{CR_\tau}$ assignments for various operators and supergravity fields are conveniently tabulated in ref. [17].

Using coordinates $x^\mu \equiv (\tau, x^1, x^2, x^3, z)$ with $L = 1$, the Euclidean black brane metric has the form

$$ds^2 = e^{2A(z)} [B(z)^2 d\tau^2 + d\mathbf{x}^2] + \frac{dz^2}{z^2}. \quad (4.1)$$

¹¹The ratio $c_{\text{I}}^{\text{bdy}} / c_{\text{II}}^{\text{bdy}}$ is fixed by the required regularity of the solution at the horizon; this ratio does not depend on the actual value of $c_{\text{II}}^{\text{bdy}}$.

We have switched to an inverted radial coordinate $z \equiv r_h/r$ which places the horizon at $z = 1$ and the boundary at $z = 0$. At the horizon, $B(z)$ vanishes while $A(z)$ remains finite and non-zero. Asymptotically, $A(z) \sim -\ln z$ and $B(z) \sim 1$ as $z \rightarrow 0$. Appendix A discusses the background geometry in more detail.

We add to this background a perturbation with arbitrary radial dependence multiplying a plane wave in the x^3 direction with imaginary wavenumber κ . In other words, $g_{\mu\nu} = g_{\mu\nu}^{\text{cl}} + h_{\mu\nu}$, where $g_{\mu\nu}^{\text{cl}}$ is the background metric (4.1) and

$$h_{\mu\nu} \equiv h_{\mu\nu}(z) e^{-\kappa x^3}. \quad (4.2)$$

We similarly expand the two scalar fields present in the supergravity dual, writing $\alpha = \alpha_{\text{cl}} + \tilde{\alpha}$ and $\chi = \chi_{\text{cl}} + \tilde{\chi}$, where the classical parts solve their respective Klein-Gordon equations [with potential (2.6)] and the perturbations, indicated by tildes, have the form

$$\tilde{\alpha} \equiv \tilde{\alpha}(z) e^{-\kappa x^3}, \quad \tilde{\chi} \equiv \tilde{\chi}(z) e^{-\kappa x^3}. \quad (4.3)$$

Under $O(2)$ rotations, the various fluctuation components transform either as scalars ($\mathcal{J} = 0$), vectors ($\mathcal{J} = 1$), or rank-2 tensors ($\mathcal{J} = 2$), as shown in table 1. The $O(2)$ symmetry guarantees that the equations for fluctuations with differing helicities decouple.

An infinitesimal diffeomorphism, $\delta x^\mu = \xi^\mu$, produces a metric perturbation $\delta h_{\mu\nu} = -\nabla_\mu \xi_\nu - \nabla_\nu \xi_\mu$, along with variations of the scalar fields given by $\delta \tilde{\alpha} = -\xi^\mu \partial_\mu \alpha_{\text{cl}}$ and $\delta \tilde{\chi} = -\xi^\mu \partial_\mu \chi_{\text{cl}}$. The covariant derivative $\nabla_\mu \xi_\nu \equiv \partial_\mu \xi_\nu - \Gamma_{\mu\nu}^\lambda \xi_\lambda$ is taken with respect to the background metric. Of particular relevance will be residual diffeomorphisms which preserve the functional form of the metric perturbation, namely $\xi_\mu \equiv \xi_\mu(z) e^{-\kappa x^3}$. It is helpful to note that there exist linear combinations of metric perturbations (without radial derivatives) which are invariant under these residual diffeomorphisms. For example, h_{12} and h_{0a} are residual diffeomorphism invariant perturbations in the tensor and vector channels, respectively. In the scalar channel, $-h_{00} + \sum_{i=1}^3 \frac{1}{2} (\Gamma_{00}^z / \Gamma_{ii}^z) h_+$ is invariant under residual diffeomorphisms. We will fix this residual diffeomorphism invariance by imposing the “axial” gauge condition,

$$h_{\mu 4} = 0, \quad (4.4)$$

at all points in space.

The Einstein equations resulting from the action (2.4) are

$$R_{\mu\nu} = 4(3\partial_\mu \alpha \partial_\nu \alpha + \partial_\mu \chi \partial_\nu \chi) + \frac{4}{3} V(\alpha, \chi) g_{\mu\nu}, \quad (4.5)$$

with the potential $V(\alpha, \chi)$ given by eq. (2.6). The scalar field equations are

$$\frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu\nu} \partial_\nu \alpha) = \frac{1}{6} V_\alpha, \quad \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu\nu} \partial_\nu \chi) = \frac{1}{2} V_\chi, \quad (4.6)$$

where here and henceforth we use subscripts to denote partial derivatives of the potential, $V_\alpha \equiv \delta V / \delta \alpha$, $V_\chi \equiv \delta V / \delta \chi$, etc. Linearizing the Einstein equations about the background

\mathcal{J}	mode	fields
2	tensor	$h_{ab} - \frac{1}{2}h_+\delta_{ab}$
1	vector	h_{0a}, h_{a3}, h_{a4}
0	scalar	$h_{00}, h_{03}, h_{04}, h_+, h_{33}, h_{34}, h_{44}, \tilde{\alpha}, \tilde{\chi}$

Table 1. Classifications of supergravity fields under $O(2)$ spatial rotations. The index $a = 1, 2$ labels transverse spatial directions, 3 is the longitudinal spatial direction, and 4 the radial direction; $h_+ \equiv h_{11} + h_{22}$.

metric $g_{\mu\nu}^{\text{cl}}$ produces the small fluctuation equation

$$\begin{aligned}
& -\frac{1}{2}\nabla_\mu\nabla_\nu h - \frac{1}{2}\nabla^2 h_{\mu\nu} + \frac{1}{2}\nabla^\lambda(\nabla_\mu h_{\nu\lambda} + \nabla_\nu h_{\mu\lambda}) \\
& = \frac{4}{3}(Vh_{\mu\nu} + V_\alpha g_{\mu\nu}^{\text{cl}}\tilde{\alpha} + V_\chi g_{\mu\nu}^{\text{cl}}\tilde{\chi}) + 12(\partial_\mu\alpha_{\text{cl}}\partial_\nu\tilde{\alpha} + \partial_\mu\tilde{\alpha}\partial_\nu\alpha_{\text{cl}}) \\
& \quad + 4(\partial_\mu\chi_{\text{cl}}\partial_\nu\tilde{\chi} + \partial_\mu\tilde{\chi}\partial_\nu\chi_{\text{cl}}). \tag{4.7}
\end{aligned}$$

The linearized scalar fluctuation equations resulting from eq. (4.6) are

$$\frac{1}{\sqrt{g_{\text{cl}}}}\partial_\mu[\sqrt{g_{\text{cl}}}(\partial^\mu\tilde{\alpha} - h^{\mu\nu}\partial_\nu\alpha_{\text{cl}})] + \frac{1}{2}\partial_\mu h\partial^\mu\alpha_{\text{cl}} = \frac{1}{6}(V_{\alpha\alpha}\tilde{\alpha} + V_{\alpha\chi}\tilde{\chi}), \tag{4.8a}$$

$$\frac{1}{\sqrt{g_{\text{cl}}}}\partial_\mu[\sqrt{g_{\text{cl}}}(\partial^\mu\tilde{\chi} - h^{\mu\nu}\partial_\nu\chi_{\text{cl}})] + \frac{1}{2}\partial_\mu h\partial^\mu\chi_{\text{cl}} = \frac{1}{2}(V_{\chi\chi}\tilde{\chi} + V_{\chi\alpha}\tilde{\alpha}). \tag{4.8b}$$

In these fluctuation equations, the background metric $g_{\mu\nu}^{\text{cl}}$ is used to raise and lower indices, compute covariant derivatives, and define the trace $h \equiv g_{\text{cl}}^{\mu\nu}h_{\mu\nu}$. Scalar potentials are evaluated on the classical solutions α_{cl} and χ_{cl} .

Not counting the dilaton and axion, there are two equations for the scalars α and χ and fifteen equations for the components of the metric (which is a symmetric tensor). In the axial gauge (4.4), five of these equations become constraints and there are actually only five independent components of the metric, which split according to helicity into two for the tensor channel, two in the vector channel, and one in the scalar channel — which couples to the scalar fields α and χ .

Tensor channel

The tensor mode involves the symmetric traceless perturbation $h_{ab} - \frac{1}{2}h_+\delta_{ab}$. Its two independent components are $h_- \equiv h_{11} - h_{22}$ and h_{12} , which obey identical second order ODEs. The resulting equation may be written more compactly if we let $b(z) \equiv \ln B(z)$,¹² and use redefined fields $H_- \equiv e^{-2A}h_-$ and $H_{12} \equiv e^{-2A}h_{12}$. The result is

$$H_-'' + (4A' + b' + z^{-1})H_-' + \kappa^2 z^{-2}e^{-2A}H_- = 0, \tag{4.9}$$

¹²This redefinition is not helpful for numerical calculations, since $b(z)$ diverges at the horizon. Nevertheless, we use it here to make the equations more concise.

(and likewise for H_{12}). This is the same equation obeyed by the dilaton and axion — they are ‘inert’ scalars which have constant background values and obey source-free Klein-Gordon equations [51]. Linearly-independent near-horizon solutions have the form

$$H_-^{\text{I}}(z) \sim 1 + \cdots, \quad H_-^{\text{II}}(z) \sim (1 + \cdots) + H_-^{\text{I}}(z) \ln(1-z), \quad (4.10)$$

where ellipses denote ascending power series in $z-1$. As anticipated, the second solution has a logarithmic singularity at the horizon, and must be discarded. Since e^{-2A}/z^2 is regular at the horizon, it follows that the appropriate boundary conditions at the horizon are $H_-(1) = c_0$ and $H'_-(1) = 0$, with c_0 an arbitrary constant. Linearly-independent near-boundary solutions have the form

$$H_-^{\text{I}}(z) \sim z^4(1 + \cdots), \quad H_-^{\text{II}}(z) \sim (1 + \cdots) + (\text{const.}) H_-^{\text{I}}(z) \ln z, \quad (4.11)$$

where these ellipses denote power series in z . The coefficient of the non-normalizable solution H_-^{II} must be set to zero, so that H_- vanishes as z^4 for small z . The numerical analysis of the resulting boundary value problem is discussed in Appendix C. The lowest screening mass obtained from this analysis is plotted in figure 6 as a function of the ratio of the mass deformation to the temperature, m/T . As $m/T \rightarrow 0$, we recover the $\mathcal{N}=4$ value $\kappa \approx 3.4041 \pi T$. Extrapolating the low temperature data shown on the right in figure 6, we find $\kappa/(\pi T) \rightarrow 3.248(5)$ as $T/m \rightarrow 0$.¹³

Vector channel

In axial gauge the only independent vector fluctuation is h_{0a} .¹⁴ The $0a$ equations are fully decoupled. After defining $H_{0a} \equiv e^{-2A} h_{0a}$, these equations take the form

$$H_{0a}'' + (4A' - b' + z^{-1})H_{0a}' + \kappa^2 z^{-2} e^{-2A} H_{0a} = 0. \quad (4.12)$$

Near the horizon, linearly independent solutions are

$$H_{0a}^{\text{I}}(z) \sim (z-1)^2(1 + \cdots), \quad H_{0a}^{\text{II}}(z) \sim (1 + \cdots) + (\text{const.}) H_{0a}^{\text{I}}(z) \ln(1-z). \quad (4.13)$$

Both solutions are finite and once differentiable at $z = 1$. Consequently, one cannot select a regular solution by specifying a boundary value and first derivative precisely at the horizon.

¹³ In performing the zero temperature extrapolation, we assume that the screening mass admits a Taylor expansion of the form $\sum_{n \geq 0} p_n (T/m)^n$. Truncating at quadratic order, we used a least-squares fit to the data starting from the smallest computed value of T/m . We looked for plateaus in plots of the coefficients p_n versus the number of fitted points, and found a very stable fit using 125 points corresponding to $0.0300 < T/m < 0.0315$. The leading coefficient $p_0 \approx 3.2484$. Deviations were scattered randomly around zero with displacements on the order of $\pm 10^{-8}$. Our estimate of the uncertainty in the last digit is based on linear and cubic fits, which fit the data more poorly than a quadratic fit and predict values for p_0 which under- and over-shoot 3.2484.

¹⁴ The h_{a4} component is zero from our gauge choice and the $(a4)$ equations become a constraint that can be solved as $h_{a3}(z) = C_{a4} e^{2A}$ for arbitrary constants C_{a4} . Using the background equations of motion (A.4), it is straightforward to show that this solution satisfies the second order $(a3)$ equations. The undetermined constant in h_{a3} reflects the fact that the axial gauge condition does not eliminate all diffeomorphism freedom.

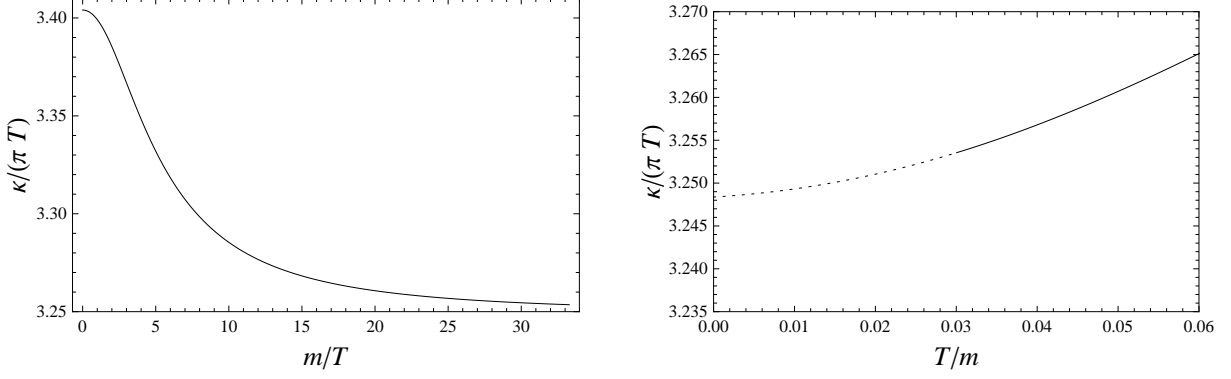


Figure 6. Screening mass (in units of πT) in the tensor channel, $\mathcal{J}^{CR_\tau} = 2^{++}$, as a function of m/T (left) and, for low temperatures, replotted as a function of T/m (right). The dotted line is an extrapolation based on a quadratic fit to the data (see footnote 13). Fluctuations of the axion field, with $\mathcal{J}^{CR_\tau} = 0_-^{+-}$, and the dilaton, with $\mathcal{J}^{CR_\tau} = 0_+^{++}$, obey exactly the same linear equation as the $\mathcal{J} = 2$ tensor mode, and consequently exhibit the same screening mass.

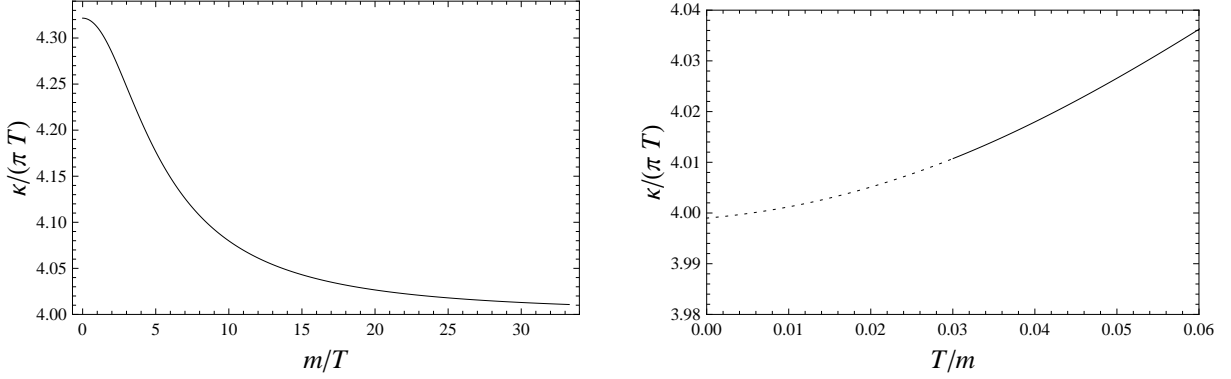


Figure 7. Screening mass (in units of πT) for the vector channel, $\mathcal{J}^{CR_\tau} = 1^{+-}$, plotted as a function of m/T (left) and, for low temperatures, replotted as a function of T/m (right). The dotted line is an extrapolation based on a quadratic fit to the data.

However, one may use the regular near-horizon asymptotic solution H_{0a}^I to generate boundary conditions at some value of z close to, but slightly less than 1.¹⁵ The near-boundary limit of the mode H_{0a} is identical to that of H_- since $b' \rightarrow 0 + O(z^3)$. Once again, we demand that H_{0a} vanish like z^4 as $z \rightarrow 0^+$. Numerical implementation of this boundary value problem is identical to that for the tensor mode. Our results for the resulting vector channel screening mass are plotted in figure 7. The limiting value for $m/T = 0$ is $\kappa \approx 4.3215 \pi T$.¹⁶ Extrapolating to zero temperature, we estimate that $\kappa/(\pi T) \rightarrow 3.999(9)$ as $T/m \rightarrow 0$.

¹⁵Alternatively, one could remove the ambiguity by redefining $H_{0a}(z) \equiv (z-1)\tilde{H}_{0a}(z)$. This shifts the roots of the indicial equation down by one unit, from $\{2, 0\}$ to $\{1, -1\}$, so that the regular solution now behaves as $\tilde{H}_{0a}(z) \sim z - 1$, giving boundary conditions $\tilde{H}_{0a}(1) = 0$ and $\tilde{H}'_{0a}(1) = 1$, while the irregular solution diverges.

¹⁶This value is slightly smaller than the $\mathcal{N} = 4$ value quoted in ref. [17] which was obtained in ref. [49]. The discrepancy of $\approx 0.0002 \pi T$ presumably reflects less accurate numerical work in these references.

Scalar channel

In axial gauge, the scalar channel couples the metric perturbations h_{00} , $h_+ \equiv h_{11} + h_{22}$, and h_{33} with the scalar perturbations $\tilde{\alpha}$ and $\tilde{\chi}$.¹⁷ The mixing of these fluctuations at nonzero m/T is inevitable — the conformal Ward identity given in eq. (A.31) involves the thermal averages of the operators \mathcal{O}_2 (dual to $\tilde{\alpha}$), \mathcal{O}_3 (dual to $\tilde{\chi}$), and the trace T^μ_μ (dual to a linear combination of h_{00} , h_+ , and h_{33}). It is convenient to make the following field redefinitions. First, let $h_+ \equiv e^{2A} H_+$ and $h_{00} \equiv e^{2(A+b)} H_{00}$. Second, define

$$Z \equiv H_{00} - \frac{1}{2}(1 + b'/A') H_+, \quad a \equiv \tilde{\alpha} - \frac{1}{4}(\alpha'_{\text{cl}}/A') H_+, \quad c \equiv \tilde{\chi} - \frac{1}{4}(\chi'_{\text{cl}}/A') H_+. \quad (4.14)$$

These fields are invariant under infinitesimal diffeomorphisms preserving axial gauge. In Appendix C we derive the coupled system of ODEs for these gauge-invariant variables. The result is

$$Z'' + pZ' + qZ = f_a a + f_c c, \quad (4.15a)$$

$$a'' + p_a a' + q_a a = g_Z(Z' + b'Z) + g_c c, \quad (4.15b)$$

$$c'' + p_c c' + q_c c = h_Z(Z' + b'Z) + h_a a, \quad (4.15c)$$

where

$$p = 4A' + b' + z^{-1} - \frac{8}{3}b'V/\mathcal{Q}, \quad f_a = \frac{8}{3}b'[(3A' + b')V_\alpha + 24\alpha'_{\text{cl}}V]/\mathcal{Q}, \quad (4.16a)$$

$$q = \kappa^2 z^{-2} e^{-2A} - \frac{8}{3}b'^2 V/\mathcal{Q}, \quad f_c = \frac{8}{3}b'[(3A' + b')V_\chi + 8\chi'_{\text{cl}}V]/\mathcal{Q}, \quad (4.16b)$$

$$p_a = 4A' + b' + z^{-1}, \quad g_Z = (\frac{1}{6}A'V_\alpha + \frac{4}{3}\alpha'_{\text{cl}}V)/\mathcal{Q}, \quad (4.16c)$$

$$p_c = 4A' + b' + z^{-1}, \quad h_Z = (\frac{1}{2}A'V_\chi + \frac{4}{3}\chi'_{\text{cl}}V)/\mathcal{Q}, \quad (4.16d)$$

$$q_a = \kappa^2 z^{-2} e^{-2A} - \frac{1}{6}z^{-2}V_{\alpha\alpha} - \frac{4}{3}[(6A' + b')\alpha'_{\text{cl}}V_\alpha + 24\alpha'^2_{\text{cl}}V]/\mathcal{Q}, \quad (4.16e)$$

$$q_c = \kappa^2 z^{-2} e^{-2A} - \frac{1}{2}z^{-2}V_{\chi\chi} - \frac{4}{3}[(6A' + b')\chi'_{\text{cl}}V_\chi + 8\chi'^2_{\text{cl}}V]/\mathcal{Q}, \quad (4.16f)$$

$$g_c = \frac{1}{6}z^{-2}V_{\alpha\chi} + \frac{4}{3}[(3A' + b')\alpha'_{\text{cl}}V_\chi + A'\chi'_{\text{cl}}V_\alpha + 8\alpha'_{\text{cl}}\chi'_{\text{cl}}V]/\mathcal{Q}, \quad (4.16g)$$

$$h_a = \frac{1}{2}z^{-2}V_{\alpha\chi} + \frac{4}{3}[(3A' + b')\chi'_{\text{cl}}V_\alpha + 9A'\alpha'_{\text{cl}}V_\chi + 24\alpha'_{\text{cl}}\chi'_{\text{cl}}V]/\mathcal{Q}, \quad (4.16h)$$

with $\mathcal{Q} \equiv z^2 A'(3A' + b')$.¹⁸

It is instructive to first consider the massless limit, $m = 0$, corresponding to the $\mathcal{N} = 4$ theory where we know that $\alpha_{\text{cl}} = \chi_{\text{cl}} = 0$. This implies that $V = -\frac{3}{4}\hat{g}^2 = -3$, $V_\alpha = V_\chi = 0$, $V_{\alpha\alpha} = -6\hat{g}^2 = -24$, $V_{\chi\chi} = -\frac{3}{2}\hat{g}^2 = -6$, and $V_{\alpha\chi} = 0$. All of the source terms on the right hand sides of eqs. (4.15) vanish, leaving fully decoupled ODEs. The scalar fields a and c obey

¹⁷The perturbation h_{03} may be found by integrating the first order (04) equation, leading to $h_{03}(z) = C_{03}B^2e^{2A}$ for an arbitrary constant C_{03} . Using the background equations of motion (A.4), one may check that this solution also satisfies the second order (03) equation.

¹⁸Equations (4.15)-(4.16) are symmetric under the interchange $\sqrt{3}\alpha \leftrightarrow \chi$. The factor of $\sqrt{3}$ reflects the normalization of the scalar kinetic terms in the action (2.5).

an equation of the form $\phi'' + (4A' + b' + 1/z)\phi' + (\kappa^2 e^{-2A}/z^2 - M^2/z^2)\phi = 0$, with $M^2 = -4$ and -3 , respectively. The leading behavior of linearly independent solutions is

$$\text{near horizon:} \quad a^{\text{I}}(z) \sim 1 + \dots, \quad a^{\text{II}}(z) \sim (1 + \dots) + a^{\text{I}}(z) \ln(1-z), \quad (4.17a)$$

$$c^{\text{I}}(z) \sim 1 + \dots, \quad c^{\text{II}}(z) \sim (1 + \dots) + c^{\text{I}}(z) \ln(1-z), \quad (4.17b)$$

$$\text{near boundary:} \quad a^{\text{I}}(z) \sim z^2(1 + \dots), \quad a^{\text{II}}(z) \sim z^2(1 + \dots) + a^{\text{I}}(z) \ln z, \quad (4.17c)$$

$$c^{\text{I}}(z) \sim z^3(1 + \dots), \quad c^{\text{II}}(z) \sim z(1 + \dots) + (\text{const.}) c^{\text{I}}(z) \ln z. \quad (4.17d)$$

The term $-M^2/z^2$ in the ODE is unimportant near $z = 1$, so both scalars have the same near-horizon behavior. The metric perturbation field Z obeys $Z'' + (4A' + b' + z^{-1} + \frac{8b'}{2+(zA')^2})Z' + (\kappa^2 z^{-2} e^{-2A} + \frac{8b'^2}{2+(zA')^2})Z = 0$. The linearly independent solutions are

$$\text{near horizon:} \quad Z^{\text{I}}(z) \sim \frac{1 + \dots}{(z-1)^2}, \quad Z^{\text{II}}(z) \sim \frac{1 + \dots}{(z-1)^2} + Z^{\text{I}}(z) \ln(1-z), \quad (4.18a)$$

$$\text{near boundary:} \quad Z^{\text{I}}(z) \sim z^4(1 + \dots), \quad Z^{\text{II}}(z) \sim (1 + \dots) + (\text{const.}) Z^{\text{I}}(z) \ln z. \quad (4.18b)$$

For numerical evaluations (at arbitrary m/T), it is convenient to make field redefinitions such that the unwanted solution (or its first derivative) diverges at the horizon and boundary, while the physical solution remains regular. Examining eqs. (4.17) and (4.18), one sees that this can be accomplished by letting

$$Z \equiv z^3(z-1)^{-2} \tilde{Z}, \quad a \equiv z^2 \tilde{a}, \quad c \equiv z^2 \tilde{c}. \quad (4.19)$$

It is also convenient to define the vector

$$\mathbf{W}(z) \equiv (\tilde{Z}', \tilde{a}', \tilde{c}', \tilde{Z}, \tilde{a}, \tilde{c}), \quad (4.20)$$

in terms of which eqs. (4.15) become a system of six first order ODEs. One can write this homogeneous system as a matrix equation of the form

$$\mathbf{W}' = \mathbf{L}(z; \kappa^2) \mathbf{W}, \quad (4.21)$$

where \mathbf{L} is a 6×6 matrix involving the coefficients (4.16) and field redefinitions (4.19). For any choice of the vector \mathbf{W} at the boundary, the solution $\mathbf{W}(z)$ at an arbitrary point in the bulk is linearly related, $\mathbf{W}(z) = \mathbf{\Phi}(z; \kappa^2) \mathbf{W}(0)$, where the transfer matrix $\mathbf{\Phi}$ depends on κ^2 and equals the identity at the boundary, $\mathbf{\Phi}(0; \kappa^2) = \mathbf{1}$. Expanding the solutions in power series near the horizon and boundary, one can show that

$$\mathbf{W}(1) = (-2\tilde{Z}_0, -2\tilde{a}_0, -2\tilde{c}_0, \tilde{Z}_0, \tilde{a}_0, \tilde{c}_0), \quad (4.22a)$$

$$\mathbf{W}(0) = (\tilde{Z}_{0,0}, 0, \tilde{c}_{0,0}, 0, \tilde{a}_{0,0}, 0). \quad (4.22b)$$

There are three undetermined coefficients at the horizon $(\tilde{Z}_0, \tilde{a}_0, \tilde{c}_0)$ and three at the boundary $(\tilde{Z}_{0,0}, \tilde{a}_{0,0}, \tilde{c}_{0,0})$. In other words, at either endpoint there is a three-dimensional subspace

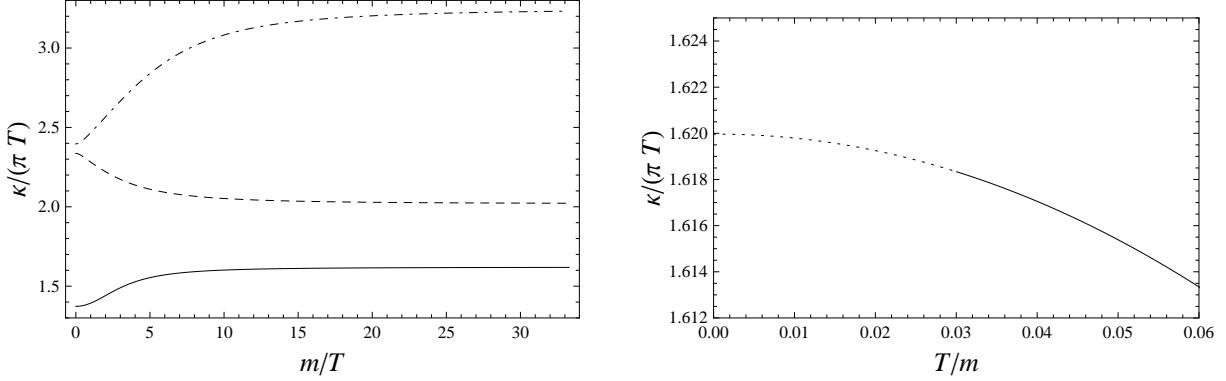


Figure 8. Screening masses (in units of πT) for the scalar channel, $\mathcal{J}_{R_y}^{CR_\tau} = 0_+^{++}$, plotted as a function of m/T (left). For low temperatures, the lightest screening mass is replotted (right) as a function of T/m . The dotted line is an extrapolation based on a quadratic fit to the data.

of solutions satisfying physical boundary conditions at that endpoint. Our task is to find those values of κ^2 for which the boundary-to-horizon transfer matrix maps some vector in the physical subspace at the boundary into a vector lying in the physical subspace at the horizon. This may be accomplished by testing linear dependence. We choose basis vectors that span the physical subspaces,

$$\text{horizon:} \quad \{(-2, 0, 0, 1, 0, 0), (0, -2, 0, 0, 1, 0), (0, 0, -2, 0, 0, 1)\}, \quad (4.23a)$$

$$\text{boundary:} \quad \{(1, 0, 0, 0, 0, 0), (0, 0, 0, 0, 1, 0), (0, 0, 1, 0, 0, 0)\}. \quad (4.23b)$$

Each basis vector defines a set of initial conditions which may be used to integrate the differential equation from one endpoint, either $z = 0$ or 1 , to a matching point in the bulk, $z = z_*$. This yields a set of six vectors $\{\mathbf{W}(z_*)_{i=1,\dots,6}\}$, where the first three are obtained by integrating from the horizon and the last three are obtained by integrating from the boundary. For generic values of κ^2 , these six vectors will be linearly independent. But if, for some value of κ^2 , a solution exists which satisfies the boundary conditions at both boundary and horizon, then the solution may be expressed either as a linear combination of $\{\mathbf{W}(z_*)_{i=1,2,3}\}$, or as a linear combination of $\{\mathbf{W}(z_*)_{i=4,5,6}\}$. In other words, for this value of κ^2 , the six vectors $\{\mathbf{W}(z_*)_{i=1,\dots,6}\}$ will not be linearly independent. If we regard these vectors as columns of a 6×6 matrix,

$$\mathbf{D} \equiv \left(\mathbf{W}(z_*)_1, \dots, \mathbf{W}(z_*)_6 \right), \quad (4.24)$$

then $\det \mathbf{D}$ must vanish for values of κ^2 which correspond to screening masses. Consequently, the task reduces to finding the roots of $\det \mathbf{D}$ as a function of κ^2 . Appendix C describes our numerical procedures in somewhat more detail.

Our results for the lowest screening masses are shown in figure 8. The operators dual to the scalar mode include the energy density T_{00} as well as \mathcal{O}_2 and \mathcal{O}_3 . When $m/T = 0$, the energy-momentum tensor and the scalar operators decouple. At this point, the values shown in figure 8 correspond, in ascending order, to the lowest screening mass of the \mathcal{O}_2 ,

T_{00} , and \mathcal{O}_3 operators in the undeformed $\mathcal{N}=4$ theory. The value of the second screening mass, $\kappa \approx 2.3361 \pi T$, associated with T_{00} agrees with previous results [17]. For any nonzero value of m/T , these operators mix. Diagonalizing the matrix of correlators will, at long distances, yield decaying exponentials corresponding to the different screening masses shown on the left in figure 8. These curves may be regarded as the beginning of a tower of possible screening masses. The lightest screening mass is shown as a solid line, and its low temperature behavior is replotted as a function of T/m on the right side of figure 8. Extrapolating to zero temperature, we estimate that $\kappa/(\pi T) \rightarrow 1.620(2)$ as $T/m \rightarrow 0$.

5 Discussion

Based on our results of screening masses in $\mathcal{N}=2^*$ SYM plasma, we make the following observations:

- The smallest screening mass occurs in the 0_+^{++} channel for any value of m/T . Fluctuations in the plasma with these quantum numbers are sourced by the energy density T_{00} , as well as the scalar operators \mathcal{O}_2 and \mathcal{O}_3 . These operators couple to the time-time component of the graviton and to quanta of the scalar fields α and χ . The associated screening mass is given by the solid curve in figure 8 which rises monotonically (in m/T) from $1.3731 \pi T$ at high temperatures to $1.620 \pi T$ at low temperatures. This screening mass should represent the mass gap, κ_{gap} , whose inverse characterizes the longest possible correlations in the plasma.¹⁹
- The smallest screening mass in an R_τ -odd channel occurs in the 0_-^{+-} channel for any value of m/T . It is shown in figure 6 and is our best candidate for the Debye mass. The associated supergravity mode is the 5D axion and its dual SYM operator is $\text{tr } E \cdot B$.
- Unlike the tensor and vector channels, the lightest screening mass in the scalar channel increases as the temperature decreases. The mass gap in the weakly coupled theory exhibits a similar behavior in the range of temperatures $m \gg T \gg \Lambda$. The mass gap in units of the temperature depends quadratically on the coupling constant $m_{\text{gap}} \sim g^2(T) T$, and for temperatures $m \gg T$ the running of $g(T)$ is the same as for an asymptotically free theory.
- The screening mass extracted from the long-distance limit of the T_{00} correlator jumps discontinuously when m/T goes from zero to an infinitesimal positive number. This happens for the following reason. The Euclidean two point function can be expressed as $\langle T_{00}(\mathbf{x}) T_{00}(0) \rangle = \sum_n e^{-\kappa_n |\mathbf{x}|} |c_n|^2$, where $\{\kappa_n\}$ may be thought of as excitation energies for the eigenstates $\{|n\rangle\}$ of the Hamiltonian (or transfer matrix) defined on $\mathbb{R}^2 \times S^1$, where the circumference of the compact spatial direction is $\beta = 1/T$, and $c_n \equiv \langle n | T_{00} | 0 \rangle$.

¹⁹To settle the question of whether this really is the gap, one should calculate the lightest screening masses in all possible symmetry channels. We have not considered SYM operators coupling to the NSNS and RR two-forms, since they have no simple 5D counterpart.

In the case of $\mathcal{N}=4$ SYM, the lowest energy with a nonvanishing amplitude to couple to T^{00} is $\kappa_0 \approx 2.3361 \pi T$ [17]. However, when a mass deformation is turned on, the $SO(6)_R$ symmetry of the massless theory is explicitly broken to an $SU(2)_R$ subgroup. Amplitudes c_n which vanish at $m=0$, due to the $SO(6)$ symmetry, may become non-vanishing at $m \neq 0$, when the R -symmetry is reduced. From figure 8 we learn that one of the corresponding energies is smaller than κ_0 . Thus, the behavior of the correlator in the $m/T \rightarrow 0$ and $|\mathbf{x}| \rightarrow \infty$ limits depends on the order of limits. Examining the long-distance behavior first, for non-zero m/T , yields a T_{00} screening mass which approaches $1.3731 \pi T$ as $m/T \rightarrow 0$.²⁰

Let us comment on similarities and differences between $\mathcal{N}=4$ SYM, $\mathcal{N}=2^*$ SYM, and QCD plasmas:

- In QCD, the mass gap and the Debye mass are found in the 0_+^{++} and 0_-^{+-} channels, respectively. This is also true for our candidate mass gap and Debye mass in $\mathcal{N}=2^*$ SYM. And it is true for $\mathcal{N}=4$ SYM with an important caveat: one must limit the comparison of operators to the R -singlet sector. It is known that the screening mass for an R -current in $\mathcal{N}=4$ SYM is less than that for the energy density [18]. However, the R -current transforms nontrivially under the $SO(6)_R$ of $\mathcal{N}=4$ SYM. QCD, of course, does not have such a symmetry, which is why the authors of ref. [17] chose to restrict comparisons to the R -singlet sector.
- In $\mathcal{N}=4$ SYM, conformal invariance implies that screening masses are always proportional to the temperature. In pure Yang-Mills theory at finite temperature, screening masses divided by the temperature are nearly constant for $1.5T_c \leq T \lesssim 4T_c$ [19, 20]. Therefore, maximally supersymmetric and non-supersymmetric non-Abelian plasmas are likely to be most similar in this temperature window. In $\mathcal{N}=2^*$ SYM, there are two regimes of m/T where screening masses scale linearly with temperature: very high temperature, $m/T \ll 1$, where the mass deformation is negligible, and asymptotically low temperature, $m/T \rightarrow \infty$. These plateaus are clearly visible in, for example, figure 8. One might expect the low temperature $\mathcal{N}=2^*$ plasma to be most similar to QCD plasma at $T \approx 2T_c$, since the heavy $\mathcal{N}=2$ matter decouples in the low temperature regime of $\mathcal{N}=2^*$ theory.

In table 2, we show screening masses in various symmetry channels, divided by πT , in QCD [$SU(3)$, $N_f=2$, $T \approx 2T_c$] [21], $\mathcal{N}=2^*$ SYM in the large m/T limit,²¹ and $\mathcal{N}=4$ SYM. As one sees from these results, screening masses in QCD at $T \approx 2T_c$ are not small compared to πT . This is a clear sign that QCD plasma in this regime is not weakly coupled. Screening masses in $\mathcal{N}=4$ SYM are roughly twice as large as in $N_f=2$ QCD at $T \approx 2T_c$. Larger screening masses (or shorter screening lengths) in $\mathcal{N}=4$, relative to QCD, is to be expected since there are many more fields contributing to screening in

²⁰We thank Andreas Karch for discussions about this point.

²¹More precisely, we show data from our largest mass value, $m/T \approx 33.3$.

$\mathcal{J}_{R_y}^{CR_\tau}$	$N_f = 2$ QCD	$\mathcal{N} = 2^*$ SYM	$\mathcal{N} = 4$ SYM
0_+^{++}	1.25(2)	1.62	2.34
0_-^{+-}	1.80(4)	3.25	3.40
1^{+-}	2.88(12)	4.01	4.32
2^{++}	2.56(7)	3.25	3.40

Table 2. Screening masses in selected symmetry channels, in units of πT , in QCD ($N_f = 2$, $T \approx 2T_c$), the large mass regime of $\mathcal{N} = 2^*$ SYM ($m/T \approx 33.3$), and $\mathcal{N} = 4$ SYM.

$\mathcal{N} = 4$ theory. As seen in table 2, screening masses in the mass-deformed $\mathcal{N} = 2^*$ theory, at large m/T , are smaller than in $\mathcal{N} = 4$, and closer to the QCD values. The decrease in screening masses, relative to $\mathcal{N} = 4$, is relatively modest except for the 0_+^{++} scalar channel, where the change is substantial.

We remind the reader that the 0_-^{+-} and 2^{++} screening masses are necessarily identical in both $\mathcal{N} = 4$ and $\mathcal{N} = 2^*$ SYM. This reflects the fact that in both these theories the axion obeys the same equation of motion as the transverse components of the graviton. A holographic theory in which these two supergravity modes obey distinct equations will be needed to more closely model QCD.

- It is also interesting to compare ratios of screening masses in different symmetry channels, instead of their absolute values. Figure 9 plots screening mass ratios, relative to the mass gap (or screening mass in the 0_+^{++} channel) as a function of m/T in $\mathcal{N} = 2^*$ SYM. Table 3 shows the ratios of screening masses relative to the mass gap for QCD [$N_f = 2$, $T \approx 2T_c$] [21], $\mathcal{N} = 2^*$ SYM in the large m/T limit, and $\mathcal{N} = 4$ SYM. The ratio of the 0_-^{+-} and 0_+^{++} screening masses is virtually identical in QCD and $\mathcal{N} = 4$ SYM. This near-perfect agreement is surely fortuitous, and in this particular case the $\mathcal{N} = 2^*$ ratio deviates much farther from the QCD value. But, as shown in table 3, in the other symmetry channels for which we have results, the $\mathcal{N} = 2^*$ screening mass ratios are much closer to the QCD values than the $\mathcal{N} = 4$ ratios.

Overall, considering both the absolute values and the ratios of screening masses in various symmetry channels, it seems fair to regard $\mathcal{N} = 2^*$ SYM as a better model of a QCD plasma than $\mathcal{N} = 4$ SYM.

- In pure Yang-Mills theory, screening masses divided by the temperature drop precipitously (at least in certain $\mathcal{J} = 0$ channels) when the temperature falls below $1.5T_c$ and approaches T_c [20]. Neither $\mathcal{N} = 4$ nor $\mathcal{N} = 2^*$ SYM have thermal phase transitions at a non-zero temperature. (For the former this is certain, and for the latter we find no evidence.) Thus, it is not sensible to ask how well either supersymmetric theory models QCD near T_c . However, as in QCD, screening masses of $\mathcal{N} = 2^*$ SYM change substan-

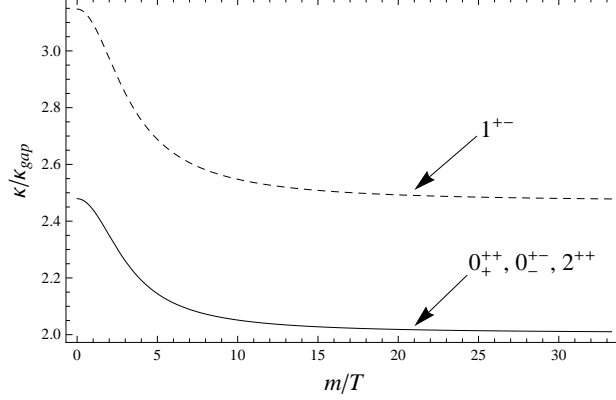


Figure 9. Ratios of $\mathcal{N}=2^*$ screening masses (in indicated channels) relative to the mass gap (or screening mass in the 0_{+}^{++} channel), as a function of m/T .

$\mathcal{J}_{R_y}^{CR_\tau}$	$N_f = 2$ QCD	$\mathcal{N}=2^*$ SYM	$\mathcal{N}=4$ SYM
0_{-}^{+-}	1.45(2)	2.01	1.46
1^{+-}	2.31(10)	2.48	1.85
2^{++}	2.05(6)	2.01	1.46

Table 3. Ratios of screening masses in the indicated symmetry channel to the mass gap (or screening mass in the 0_{+}^{++} channel), in QCD ($N_f = 2$, $T \approx 2T_c$), the large mass regime of $\mathcal{N}=2^*$ SYM ($m/T \approx 33.3$), and $\mathcal{N}=4$ SYM.

tially over a relatively small temperature range. From figures 6–9, it is clear that the high temperature plateau is limited to $m/T \lesssim 1$ and the low temperature plateau is roughly $m/T \gtrsim 10$ — significant variation is confined to the band $m/T \approx \text{few}$. The point $m/T \approx 4.83$, where the trace anomaly deviates maximally from zero (c.f. figure 4), lies in the middle of this band.

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A Properties of the gravity dual

In this and the following appendices, we present details of our analytical and numerical calculations at a level which would allow someone, with some effort, to reproduce the results

presented above. Further details, including explicit expressions for the coefficients of series expansions and more explicit derivations, may be obtained online [30].

This appendix summarizes the 5D $\mathcal{N}=8$ supergravity solution dual to finite temperature large- N_c $\mathcal{N}=2^*$ SYM. Most of the discussion is taken directly from the original works [24–26]. We collect the results in one place for two reasons: to explain, in detail, how our calculations were performed, and to benefit readers wishing to do further studies. We used *Mathematica* 7.0 for both analytical and numerical computations.

Equations of motion

We take the following ansatz for the 5D black brane metric,

$$ds^2 = e^{2A(r)}(-B(r)^2 dt^2 + d\mathbf{x}^2) + dr^2. \quad (\text{A.1})$$

Without loss of generality we have chosen coordinates such that $g_{rr} = 1$ everywhere. The metric is invariant under 3d spatial rotations, as well as time and spatial translations. The spacetime is defined for values of the radial coordinate $r_h \leq r < \infty$. The lower end is an event horizon with the property that $B(r_h) = 0$, and $A(r_h) = A_h$, a finite number. Infinity is a boundary where the metric approaches that of AdS_5 . This requires $\lim_{r \rightarrow \infty} B(r) = 1$ and $A(r)$ to asymptotically approach r/L .²² The scalar fields are chosen to be functions of r only. They should be finite at the horizon: $\alpha(r_h) = \alpha_h$ and $\chi(r_h) = \chi_h$, and vanish at infinity: $\lim_{r \rightarrow \infty} \alpha(r) = \lim_{r \rightarrow \infty} \chi(r) = 0$. We can shift r and compensate the change in the asymptotic form of the metric by rescaling the spacetime coordinates. Let us then choose $r_h = 0$ without loss of generality. The radial coordinate r is not well-suited to numerical calculations since the boundary is at infinity. Let us switch to a new, dimensionless radial coordinate $z \equiv e^{-r/L}$.²³ This places the horizon at $z = 1$ and the boundary at $z = 0$. In this new coordinate system, $A(z) \sim -\ln z$ as $z \rightarrow 0$. Since the logarithmic divergence is troublesome for numerical work it is helpful to extract this asymptotic behavior by defining

$$A(z) \equiv -\ln z + \tilde{A}(z). \quad (\text{A.2})$$

Note that this field redefinition does not significantly alter the behavior of the warp factor A near the horizon. In particular, $\tilde{A}(1) = A_h$. The metric now reads

$$ds^2 = \frac{e^{2\tilde{A}(z)}}{z^2}(-B(z)^2 dt^2 + d\mathbf{x}^2) + L^2 \frac{dz^2}{z^2}. \quad (\text{A.3})$$

²²The asymptotic behavior of $A(r)$ could also be $r/L + A_\infty$, where A_∞ is an arbitrary constant. However, A_∞ can be set to zero by rescaling the time coordinate.

²³Some derivations are more transparent using the r coordinate, whereas technical calculations are better handled using the z coordinate. We will freely switch back and forth between r and z in the following.

The Einstein equations derived from the action (2.4) and metric (A.3) are

$$B' = Cz^3 e^{-4\tilde{A}}, \quad (\text{A.4a})$$

$$(\tilde{A}' - 1/z)^2 + \frac{1}{2}(\tilde{A}' - 1/z)B'/B - \alpha'^2 - \frac{1}{3}\chi'^2 + \frac{L^2 V}{3z^2} = 0, \quad (\text{A.4b})$$

$$\alpha'' + (4\tilde{A}' + B'/B - 3/z)\alpha' - \frac{L^2 V_\alpha}{6z^2} = 0, \quad (\text{A.4c})$$

$$\chi'' + (4\tilde{A}' + B'/B - 3/z)\chi' - \frac{L^2 V_\chi}{2z^2} = 0, \quad (\text{A.4d})$$

where a prime denotes d/dz and C is a constant that will be determined later. The system (A.4) is sixth order.²⁴ Note that factors of L^2 appearing in the system and in the gauged supergravity coupling present in the scalar potential cancel. Therefore, it is natural to set $L = 1$ and $\hat{g}^2 = 4$.²⁵

Near-horizon solution: temperature and entropy

To have a regular black brane horizon, near $z = 1$, we try a power series solution of the form

$$B(z) = -B_h(z-1) \left[1 + \sum_{n \geq 1} B_n(z-1)^n \right], \quad (\text{A.5a})$$

$$\tilde{A}(z) = A_h + \sum_{n \geq 1} \tilde{A}_n(z-1)^n, \quad (\text{A.5b})$$

$$\alpha(z) = \alpha_h + \sum_{n \geq 1} \alpha_n(z-1)^n, \quad (\text{A.5c})$$

$$\chi(z) = \chi_h + \sum_{n \geq 1} \chi_n(z-1)^n. \quad (\text{A.5d})$$

Substituting these into system (A.4) (first taking the logarithm of eq. (A.4a) then differentiating with respect to z), we can solve the equations order-by-order in $z - 1$. We solve up to sixth order. The asymptotic values A_h , B_h , α_h , and χ_h are undetermined, and will be referred to as “horizon data.” All higher coefficients $\{B_n, \tilde{A}_n, \alpha_n, \chi_n\}$ are fully determined by the horizon data. At the horizon, the fields and their slopes are finite, as desired,

$$B(1) = 0, \quad B'(1) = -B_h, \quad \alpha(1) = \alpha_h, \quad \alpha'(1) = 0, \quad (\text{A.6a})$$

$$\tilde{A}(1) = A_h, \quad \tilde{A}'(1) = 1, \quad \chi(1) = \chi_h, \quad \chi'(1) = 0. \quad (\text{A.6b})$$

The Hawking temperature of the black brane may be extracted from the metric near the horizon using the usual analytical continuation to Euclidean signature with a compact time

²⁴By making the clever choice of radial coordinate $t(z) \equiv 1 - B(z)$, the authors of ref. [26] reduce the system to a set of three second order ODEs which does not explicitly contain B or its derivative. We decided not to use the t coordinate because it leads to near-horizon and near-boundary expansions with fractional powers of t ; we find the series solutions simpler to understand in the z coordinate.

²⁵This is different from the convention used in ref. [26] where $L = 2$ and $\hat{g}^2 = 1$.

direction, and demanding that the solution is regular at the horizon, where the time circle collapses to zero size. This fixes the period of the temporal circle to be

$$\beta = \frac{2\pi L}{e^{A_h} B_h}. \quad (\text{A.7})$$

In the AdS/CFT correspondence, we identify the black brane temperature $T = 1/\beta$ with the temperature in the quantum field theory.

The Bekenstein-Hawking entropy of the black brane is proportional to the volume of its three-dimensional horizon divided by Newton's constant in five dimensions, $S = V_{\text{horizon}}/(4G_5)$. The horizon volume is $V_{\text{horizon}} = \int_{\mathbb{R}^3} d^3x \sqrt{\det g_{ab}^{\text{ind}}}$, where $g_{ab}^{\text{ind}} = e^{2A_h} \delta_{ab}$ is the induced metric on the hypersurface $z = 1$ and fixed τ . Therefore, $V_{\text{horizon}} = \mathcal{V} e^{3A_h}$, where \mathcal{V} is the (formally infinite) volume of \mathbf{x} -space. The entropy density is

$$S/\mathcal{V} = \frac{e^{3A_h}}{4G_5} = \frac{e^{3A_h} N_c^2}{2\pi L^3} = \frac{4\pi^2 N_c^2 T^3}{B_h^3}. \quad (\text{A.8})$$

Writing this as in eq. (3.2), we identify $\sigma \equiv (2/B_h)^3$. Positivity of entropy implies that $B_h > 0$. We can fix the unknown constant C in eq. (A.4a) using the near-horizon solution. Plugging eqs. (A.5a) and (A.5b) into eq. (A.4a), then evaluating at $z = 1$, we obtain $C = -e^{4A_h} B_h$.

Near-boundary solution: gauge-gravity dictionary

Near $z = 0$, solutions may be expanded in an asymptotic series solution of the form

$$B(z) = 1 + \sum_{n=0}^{\infty} \sum_{k=0}^n B_{2n+4,k} z^{2n+4} (\ln z)^k, \quad (\text{A.9a})$$

$$\tilde{A}(z) = \sum_{n=0}^{\infty} \sum_{k=0}^{n+1} \tilde{A}_{2n+2,k} z^{2n+2} (\ln z)^k, \quad (\text{A.9b})$$

$$\alpha(z) = \sum_{n=0}^{\infty} \sum_{k=0}^{n+1} \alpha_{2n,k} z^{2n+2} (\ln z)^k, \quad (\text{A.9c})$$

$$\chi(z) = \sum_{n=0}^{\infty} \sum_{k=0}^n \chi_{2n,k} z^{2n+1} (\ln z)^k. \quad (\text{A.9d})$$

The leading (small z) behavior of the fields B and \tilde{A} are dictated by the requirement that, at the boundary, the metric approaches that of AdS_5 . Moreover, the two scalar fields must vanish at the boundary. The rate at which they vanish (i.e., the leading exponent) is determined by linearizing the scalar equations of motion at $z = 0$ and solving the characteristic equation for an ansatz of the form z^p .²⁶ The presence of $\ln z$ in the leading series solution comes from

²⁶The scalar potential is, to quadratic order in the small fields, $-L^2 V \approx 3 + 12\alpha^2 + 3\chi^2$. Therefore, the linearized scalar equations are $\alpha'' - (3/z)\alpha' + (4/z^2)\alpha = 0$ and $\chi'' - (3/z)\chi' + (3/z^2)\chi = 0$. The first equation has $p = 2$ as a double root. Therefore, a pair of linearly independent solutions is z^2 and $z^2 \ln z$. The second equation has $p = \{3, 1\}$, so a pair of linearly independent solutions is z^3 and $z + cz^3 \ln z$, with c a nonzero coefficient determined by the equations of motion.

the fact that $z = 0$ is a regular singular point and the roots of the characteristic equation differ by an integer. Higher powers of $\ln z$ arise from nonlinearities in the system (A.4). The unknown coefficients are found by plugging the ansatz into the system (A.4) and solving order-by-order in $z^a(\ln z)^b$. We carried this through up to order $a = 8$. Some coefficients vanish, $A_{2,1} = B_{6,1} = 0$, but generically all terms in the series (A.9) are non-zero. Observe that eq. (A.4a) determines the $B_{n,k}$ in terms of the $\tilde{A}_{n,k}$, and eq. (A.4b) determines the $\tilde{A}_{n,k}$ in terms of the $\alpha_{n,k}$ and $\chi_{n,k}$. The first order equations (A.4a) and (A.4b) should be thought of as constraints. The two scalar equations determine all higher $\alpha_{n,k}$ and $\chi_{n,k}$ in terms of the first couple of coefficients. All higher coefficients $\{B_{n,k}, \tilde{A}_{n,k}, \alpha_{n,k}, \chi_{n,k}\}$ are fully determined by the “boundary data”: $B_{4,0}$, $\alpha_{0,1}$, $\alpha_{0,0}$, $\chi_{0,0}$, and $\chi_{2,0}$. At the boundary, the fields and their slopes are

$$\begin{aligned} B(0) &= 1, & B'(0) &= 0, & \alpha(0) &= 0, & \alpha'(0) &= 0, \\ \tilde{A}(0) &= 0, & \tilde{A}'(0) &= 0, & \chi(0) &= 0, & \chi'(0) &= \chi_{0,0}. \end{aligned} \quad (\text{A.10})$$

We can fix the constant C in eq. (A.4a) using the near-boundary solution. In fact, we may evaluate this equation at any value of z and the constant C must be the same — doing this at the horizon and boundary provides a powerful constraint. Plugging eqs. (A.9a) and (A.9b) into eq. (A.4a), dividing by z^3 , then evaluating at $z = 0$, yields $C = 4B_{4,0}$. We now have a nontrivial equation that relates boundary and horizon data,

$$4B_{4,0} = -e^{4A_h} B_h. \quad (\text{A.11})$$

To summarize, the near-boundary solutions are

$$B(z) = 1 + B_{4,0} z^4 + O(z^6), \quad (\text{A.12a})$$

$$\tilde{A}(z) = \tilde{A}_{2,0} z^2 + O(z^4 \ln^2 z), \quad (\text{A.12b})$$

$$\alpha(z) = z^2 [\alpha_{0,1} \ln z + \alpha_{0,0} + O(z^2 \ln^2 z)], \quad (\text{A.12c})$$

$$\chi(z) = z [\chi_{0,0} + z^2 (\frac{4}{3} \chi_{0,0}^3 \ln z + \chi_{2,0}) + O(z^4 \ln^2 z)]. \quad (\text{A.12d})$$

Generally, in the AdS/CFT correspondence, the leading coefficient in a near-boundary expansion corresponds to the source for the dual operator and the subleading coefficient corresponds to the expectation value of that operator. The operators dual to α and χ are \mathcal{O}_2 and \mathcal{O}_3 , respectively. Hence $\alpha_{0,1}$ will be proportional to m^2 and $\alpha_{0,0}$ is related to the thermal expectation $\langle \mathcal{O}_2 \rangle$. Likewise, $\chi_{0,0}$ will be proportional to m and $\chi_{2,0}$ related to $\langle \mathcal{O}_3 \rangle$.

The precise correspondence relies on a matching calculation at zero temperature. At $T = 0$ the background metric must be Poincaré-invariant. This forces $B = 1$. The remaining fields A , α , and χ obey a system of 1st order ODEs referred to as the supersymmetric flow equations [23], whose solution is

$$e^A = k e^{2\alpha} / \sinh 2\chi, \quad (\text{A.13a})$$

$$e^{6\alpha} = \cosh 2\chi + (\gamma + \ln \tanh \chi) \sinh^2(2\chi), \quad (\text{A.13b})$$

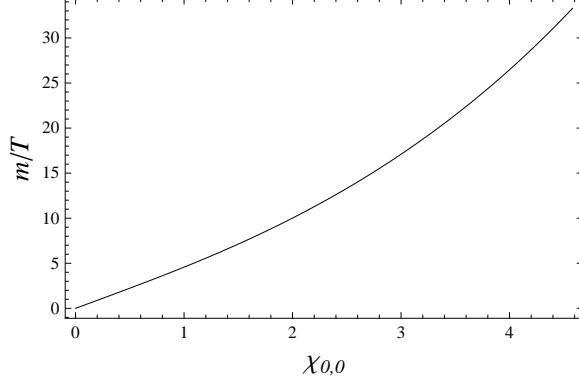


Figure 10. m/T as a function of $\chi_{0,0}$.

where k and γ are constants of integration. Plugging in the asymptotic formulas for the fields fixes the undetermined coefficients. At leading order,

$$\alpha_{0,1} = \frac{2}{3}\chi_{0,0}^2, \quad \chi_{0,0} = k/2. \quad (\text{A.14})$$

At the first subleading order one finds

$$\alpha_{0,0} = \frac{1}{3}\chi_{0,0}^2 (1 + 2\gamma + 2 \ln \chi_{0,0}), \quad \chi_{2,0} = 2\alpha_{0,0} \chi_{0,0} - \frac{1}{3}\chi_{0,0}^3. \quad (\text{A.15})$$

The constant γ parameterizes a family of distinct solutions. Solutions with $\gamma \leq 0$ correspond to points on the Coulomb branch of $\mathcal{N}=2^*$ SYM (where the scalars ϕ_1 and ϕ_2 are massive and ϕ_3 gets an expectation value). Solutions with $\gamma > 0$ are unphysical. The expressions given in eq. (A.14) for the leading series coefficients should not change at finite temperature since they are bare couplings in a Lagrange density. As such, the constant k must equal mL times a pure number. Eq. (A.14) must also hold for the black brane solution. However, eq. (A.15) holds only at zero temperature. For the case $\gamma = 0$, a probe D3-brane computation shows that the supergravity solution may be understood as a Coulomb branch vacuum in which the background D3-branes form a linear enhançon singularity. A computation of the size of the enhançon shows that $k = mL$. Thus,

$$\chi_{0,0} = \frac{mL}{2}, \quad \alpha_{0,1} = \frac{(mL)^2}{6}. \quad (\text{A.16})$$

We choose to regard $\chi_{0,0}$ as an independent variable on which the other supergravity coefficients depend. In other words, the boundary and horizon data are all functions of $\chi_{0,0}$:

$$B_{4,0}(\chi_{0,0}), \quad \alpha_{0,0}(\chi_{0,0}), \quad \chi_{2,0}(\chi_{0,0}), \quad A_h(\chi_{0,0}), \quad \alpha_h(\chi_{0,0}), \quad \chi_h(\chi_{0,0}). \quad (\text{A.17})$$

There are six independent data — $\alpha_{0,1}$ is fixed by eq. (A.16) and B_h is fixed by eq. (A.11).

Solving for L in eq. (A.7), the map from $\chi_{0,0}$ in eq. (A.16) to m/T is

$$m/T = \frac{4\pi \chi_{0,0}}{e^{A_h} B_h}. \quad (\text{A.18})$$

A plot of this map is shown in figure 10. The mapping is nonlinear and monotonic, with larger values of $\chi_{0,0}$ corresponding to larger values of m/T . Since $A_h(0)$ and $B_h(0)$ are nonzero (see Appendix B), $m/T = 0$ when $\chi_{0,0} = 0$.

Holographic renormalization, thermodynamics and expectation values

The Euclidean supergravity action is regulated and renormalized following the approach of ref. [25]. Proper renormalization is needed to define thermal expectation values correctly. We start by cutting off the boundary at the hypersurface given by the equation $r = r_0$. To the bulk Euclidean action S_{bulk} given by (minus) eq. (2.4), we add the Gibbons-Hawking boundary term

$$S_{\text{bdy}} = -\frac{1}{8\pi G_5} \int_{r=r_0} d^4x \sqrt{h} K, \quad (\text{A.19})$$

where h is the determinant of the induced metric $h_{\mu\nu}$ on the hypersurface, and K is the trace of the extrinsic curvature. The sum of these actions will diverge in the limit that $r_0 \rightarrow \infty$. To cure this one must add counterterms at the boundary. When the boundary is flat the counterterm action has the general form

$$S_{\text{ctr}} = \frac{\beta V}{4\pi G_5} e^{4A} B \left(c_1 + c_3 \alpha + c_4 \chi^2 + c_5 \alpha^2 + c_6 \alpha \chi^2 + c_8 \frac{\alpha^2}{\ln x} + c_{10} \chi^4 \ln x + c_{11} \chi^4 \right) \Big|_{r=r_0}. \quad (\text{A.20})$$

where $x \equiv \sqrt{g_{00}(r_0)} = e^{A(r_0)} B(r_0)$ parameterizes the location of the boundary. Note that the counterterm action density is completely local and that only even powers of χ appear (this is necessitated by the fact that only even powers of z appear in the near-boundary series for \tilde{A} , B , and α). By tuning the coefficients $c_{i=1,\dots,11}$ appropriately, the regulated action $S_{\text{reg}} \equiv S_{\text{bulk}} + S_{\text{bdy}} + S_{\text{ctr}}$ may remain finite upon taking the limit $r_0 \rightarrow \infty$.

The on-shell bulk action, with the Gibbons-Hawking term evaluates to

$$S_{\text{bulk}} = \frac{\beta V}{8\pi G_5} \left[e^{3A} \frac{d}{dr} (e^A B) \right]_0^{r_0} = \frac{\beta V}{8\pi G_5} \Delta - S, \quad (\text{A.21})$$

where, using eqs. (A.7) and (A.8), the term obtained from the lower limit is just the entropy S . The quantity Δ diverges in the limit $r_0 \rightarrow \infty$. Switching to the z radial coordinate (so that the hypersurface sits at $z = z_0 = e^{-r_0/L}$), we see that the leading singular behavior of Δ arises from $e^{4A(z_0)} \sim z_0^{-4}$. The counterterm coefficients are chosen to remove the sensitivity of Δ on the cutoff in the limit $z_0 \rightarrow 0$. A straightforward expansion in powers of z_0^{-1} and $\ln z_0$ leads to

$$c_1 = \frac{3}{2L}, \quad c_3 = 0, \quad c_4 = \frac{1}{L}, \quad c_5 = \frac{6}{L}, \quad c_6 = 0, \quad c_8 = -\frac{3}{L}, \quad c_{10} = -\frac{4}{3L}. \quad (\text{A.22})$$

The coefficient c_{11} multiplies a finite counterterm and corresponds to a shift in the subtraction scheme. It may be fixed unambiguously as follows. The renormalized supergravity action is identified with the gauge theory free energy divided by the temperature, $S_{\text{ren}} \equiv \lim_{z_0 \rightarrow 0} S_{\text{reg}} = \beta F$. Using $F = E - TS$, the energy is given by $E = \frac{V}{8\pi G_5} \lim_{z_0 \rightarrow 0} \Delta$. At zero temperature,

the energy must vanish since the ground state is supersymmetric. For Δ to vanish in this limit, c_{11} must equal $\frac{1}{3L}$. Ultimately,

$$\lim_{z_0 \rightarrow 0} \Delta = -\frac{3}{L} B_{4,0} - \frac{2}{L} \chi_{0,0} [\chi_{2,0} - (2\alpha_{0,0} \chi_{0,0} - \frac{1}{3} \chi_{0,0}^3)]. \quad (\text{A.23})$$

Converting from $1/L$ to T , and using eq. (A.11), the free energy density can be written as

$$F/\mathcal{V} = -\frac{\pi^2}{8} N_c^2 T^4 \left(\frac{2}{B_h} \right)^3 \left[1 - \frac{2}{B_{4,0}} \left(\chi_{2,0} \chi_{0,0} + \frac{1}{3} \chi_{0,0}^4 - 2\alpha_{0,0} \chi_{0,0}^2 \right) \right]. \quad (\text{A.24})$$

The overall factor $-\frac{\pi^2}{8} N_c^2 T^4$ is the free energy density of $\mathcal{N}=4$ SYM. The remainder of the expression constitutes the scaling function f in eq. (3.1).

In ideal fluid hydrodynamics, the energy-momentum tensor $T^{\mu\nu}$ can be expressed in terms of the local fluid four-velocity $u^\mu(\mathbf{x})$ as $T^{\mu\nu} = (\epsilon + p)u^\mu u^\nu + p g^{\mu\nu}$, where the energy density $\epsilon \equiv E/\mathcal{V}$ and the pressure $p = -\partial F/\partial V$. The trace is given by

$$\begin{aligned} -\langle T^\mu_\mu \rangle &= \epsilon - 3p = (E + 3F)/\mathcal{V} \\ &= N_c^2 T^4 \frac{\pi^2}{B_{4,0}} \left(\frac{2}{B_h} \right)^3 \left(\chi_{2,0} \chi_{0,0} + \frac{1}{3} \chi_{0,0}^4 - 2\alpha_{0,0} \chi_{0,0}^2 \right). \end{aligned} \quad (\text{A.25})$$

In a conformal theory, $\epsilon = 3p$. The factor following $N_c^2 T^4$ is the scaling function ω in eq. (3.4).

We now describe how to obtain supergravity formulas for the thermal expectation values of the operators \mathcal{O}_2 and \mathcal{O}_3 using methods described in refs. [52, 53]. These expectation values are given by functional derivatives of the renormalized supergravity action with respect to the couplings m^2 and m (which, for this discussion, should be thought of as independent couplings),

$$\langle \mathcal{O}_2 \rangle \equiv (\beta V)^{-1} \frac{\delta S_{\text{ren}}}{\delta m^2} = \lim_{r_0 \rightarrow \infty} \frac{1}{\beta V} \frac{\delta S_{\text{reg}}}{\delta \alpha(r_0)} \frac{\delta \alpha(r_0)}{\delta m^2}, \quad (\text{A.26a})$$

$$\langle \mathcal{O}_3 \rangle \equiv (\beta V)^{-1} \frac{\delta S_{\text{ren}}}{\delta m} = \lim_{r_0 \rightarrow \infty} \frac{1}{\beta V} \frac{\delta S_{\text{reg}}}{\delta \chi(r_0)} \frac{\delta \chi(r_0)}{\delta m}. \quad (\text{A.26b})$$

Let us evaluate the first functional derivative in each of the above expressions. Computing the scalar functional derivatives for S_{ctr} is straightforward with eq. (A.20), so let us focus on S_{bulk} . We have

$$\begin{aligned} S_{\text{bulk}} &= -\frac{1}{4\pi G_5} \int d^5x \sqrt{g} \left(\frac{1}{4} R + \mathcal{L}_{\text{matter}} \right) \\ &= \frac{1}{4\pi G_5} \int d^5x \sqrt{g} \left[5\dot{A}^2 + \frac{5}{2} \dot{A}\dot{B}/B + 2\ddot{A} + \frac{1}{2} \ddot{B}/B + 3\dot{\alpha}^2 + \dot{\chi}^2 + V(\alpha, \chi) \right] \\ &= \frac{1}{4\pi G_5} \int d^5x \sqrt{g} \left[\frac{1}{2} (\dot{K} + K^{\mu\nu} K_{\mu\nu}) + \frac{3}{2} (2\dot{A}^2 + \dot{A}\dot{B}/B) + 3\dot{\alpha}^2 + \dot{\chi}^2 + V(\alpha, \chi) \right] \\ &= \frac{1}{4\pi G_5} \int d^5x \sqrt{g} \left[\frac{1}{2} (-K^2 + K^{\mu\nu} K_{\mu\nu}) + 6\dot{\alpha}^2 + 2\dot{\chi}^2 \right] + \frac{1}{8\pi G_5} \int d^4x \sqrt{g} K \Big|_{r=r_0}. \end{aligned} \quad (\text{A.27})$$

The functional derivatives of the action are to be evaluated at r_0 . They should be viewed as the canonical momenta conjugate to α and χ . One finds,

$$\frac{\delta S_{\text{reg}}}{\delta \alpha(r_0)} = \frac{\beta V}{4\pi G_5} e^{4A} B \left(6\dot{\alpha} + 2c_5\alpha + 2c_8 \frac{\alpha}{\ln x} \right) \Big|_{r=r_0}, \quad (\text{A.28a})$$

$$\frac{\delta S_{\text{reg}}}{\delta \chi(r_0)} = \frac{\beta V}{4\pi G_5} e^{4A} B \left(2\dot{\chi} + 2c_4\chi + 4c_{10}\chi^3 \ln x + 4c_{11}\chi^3 \right) \Big|_{r=r_0}. \quad (\text{A.28b})$$

Continuing with the calculation, write

$$\frac{\delta \alpha(z_0)}{\delta m^2} = \frac{\delta \alpha_{0,1}}{\delta m^2} \frac{\delta \alpha(z_0)}{\delta \alpha_{0,1}} = \frac{L^2}{6} z_0^2 [\ln z_0 + O(z_0^2 \ln^2 z_0)], \quad (\text{A.29a})$$

$$\frac{\delta \chi(z_0)}{\delta m} = \frac{\delta \chi_{0,0}}{\delta m} \frac{\delta \chi(z_0)}{\delta \chi_{0,0}} = \frac{L}{2} z_0 [1 + O(z_0^2 \ln z_0)]. \quad (\text{A.29b})$$

Putting it all together and sending the cutoff to zero (it is convenient to do this in the z coordinate) yields

$$\langle \mathcal{O}_2 \rangle = \frac{N_c^2}{2\pi^2 L^2} \alpha_{0,0}, \quad \langle \mathcal{O}_3 \rangle = -\frac{N_c^2}{\pi^2 L^3} (\chi_{2,0} + \frac{1}{3} \chi_{0,0}^3). \quad (\text{A.30})$$

As expected, the subleading coefficients of the scalar fields in 5D supergravity are directly related to the expectation values of the 4d gauge theory operators to which the leading coefficients couple. Lastly, one may check that the expectation values we have computed satisfy the conformal Ward identity,

$$-\langle T^\mu_\mu \rangle = 2m^2 \langle \mathcal{O}_2 \rangle + m \langle \mathcal{O}_3 \rangle. \quad (\text{A.31})$$

At zero temperature, one may insert the value of $\chi_{2,0}$ found in eq. (A.15) and confirm that $\lim_{T \rightarrow 0} \langle T^\mu_\mu \rangle = 0$.

Endpoints of thermal scalar flows

Figure 5 showed the trend that the thermal solutions for the scalars α and χ became successively better approximations to the zero temperature enhançon solution as $m/T \rightarrow \infty$. In figure 11 we provide further evidence for this assertion by plotting the residuals to eq. (A.13), evaluated at the horizon, for our numerical backgrounds. On the left is a plot of $e^{A_h} - 2\chi_{0,0} e^{2\alpha_h} / \sinh 2\chi_h$, and on the right a plot of $e^{6\alpha_h} - [\cosh 2\chi_h + (\gamma + \ln \tanh \chi_h) \sinh^2(2\chi_h)]$ for $\gamma = 0$. In each plot the deviation from zero vanishes as we approach larger values of $\chi_{0,0}$, which corresponds to lower temperatures.

First law of thermodynamics

The first law of thermodynamics, $dE = TdS$, is equivalent to the statement that $dF/dT = -S$. Let us express this constraint in terms of supergravity parameters. For temporary convenience

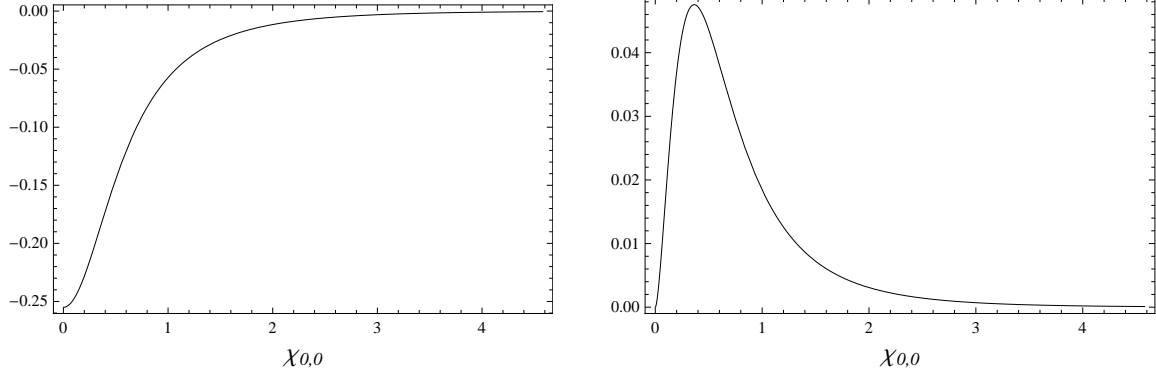


Figure 11. Deviation of endpoints of thermal flows from supersymmetric flow equations. Left: a plot of $e^{A_h} - 2\chi_{0,0}e^{2\alpha_h}/\sinh 2\chi_h$ vs. $\chi_{0,0}$. This measures how well eq. (A.13a) is satisfied at the horizon. Right: a plot of $e^{6\alpha_h} - [\cosh 2\chi_h + (\gamma + \ln \tanh \chi_h) \sinh^2(2\chi_h)]$ vs. $\chi_{0,0}$. This measures how well eq. (A.13b) is satisfied at the horizon.

define $\Omega \equiv \chi_{2,0}\chi_{0,0} + \frac{1}{3}\chi_{0,0}^4 - 2\alpha_{0,0}\chi_{0,0}^2$. Differentiating the formula for the free energy density in eq. (A.24) gives

$$\frac{dF}{dT} = -S \left\{ 1 - \frac{2\Omega}{B_{4,0}} + \frac{1}{4}B_h^3 T \frac{d}{dT} \left(\frac{1 - 2\Omega/B_{4,0}}{B_h^3} \right) \right\}. \quad (\text{A.32})$$

The entire term inside the braces must equal 1. Note that the temperature derivative acts on objects which are functions of $\chi_{0,0}$. Using the chain rule, $T \frac{d}{dT} = T \frac{d\chi_{0,0}}{dT} \frac{d}{d\chi_{0,0}}$, we need

$$T \frac{d\chi_{0,0}}{dT} = [A'_h + B'_h/B_h - 1/\chi_{0,0}]^{-1}. \quad (\text{A.33})$$

This relation comes from differentiating eq. (A.18), with primes denoting $d/d\chi_{0,0}$. After straightforward manipulation, the first law condition becomes²⁷

$$\frac{1}{2}\chi'_{2,0}\chi_{0,0} - \frac{3}{2}\chi_{2,0} - \alpha'_{0,0}\chi_{0,0}^2 + 2\alpha_{0,0}\chi_{0,0} + \frac{3}{4}B'_{4,0} - 3A'_h B_{4,0} = 0. \quad (\text{A.34})$$

This is a 1st order ODE with respect to the parameter $\chi_{0,0}$, involving four of the six independent horizon and boundary data. Therefore, it is a nontrivial constraint which serves as a robust check on our numerics [26].

The left hand side of eq. (A.34) is plotted in figure 12. While no numerical computation with finite precision numbers can ever give a result which is exactly zero, the scale on our plot indicates that the first law is satisfied accurately to within 10^{-5} for the largest $\chi_{0,0}$ (or lowest temperatures) explored. The deviation from zero is due to discretization effects.²⁸

²⁷This simple form is based on the assumption that the relation between T and $\chi_{0,0}$ is monotonic. We have verified numerically that $B'_h/B_h + A'_h \neq 1/\chi_{0,0}$. Notice that $d\chi_{0,0}/dT$ is always nonzero since B_h is always positive; this follows from the positivity of the entropy.

²⁸To demonstrate that the deviation is a discretization effect, we tried using a better approximation for the

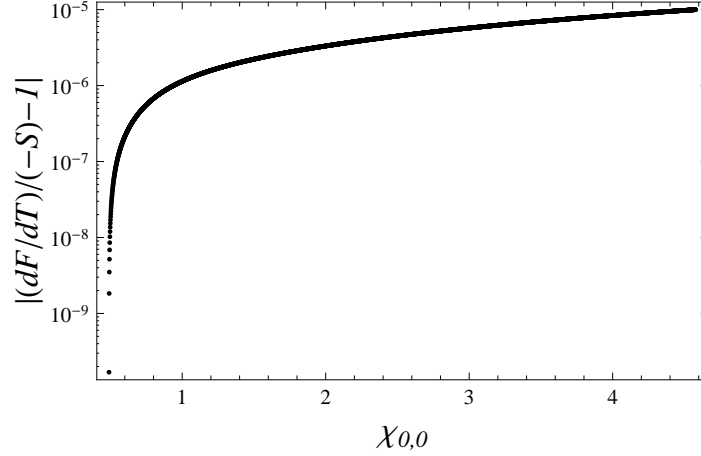


Figure 12. Deviation from first law of thermodynamics. The left hand side of eq. (A.34) is plotted versus $\chi_{0,0}$. The deviation may be attributed to discretization error.

B Numerical procedure

We used a *shooting technique* to determine the independent boundary and horizon data (A.17) as functions of $\chi_{0,0}$. The algorithm is as follows. For a given value of $\chi_{0,0}$, start with a trial set of independent boundary and horizon data,

$$\mathbf{X}(\chi_{0,0}) = (B_{4,0}, \alpha_{0,0}, \chi_{2,0}, A_h, \alpha_h, \chi_h)(\chi_{0,0}). \quad (\text{B.1})$$

The remaining parameters are fixed by $\alpha_{0,1} = \frac{2}{3}\chi_{0,0}^2$ and $B_h = -4B_{4,0}e^{-4A_h}$. One may now construct a near-horizon series solution using eq. (A.5). Use this series to evaluate

$$\mathbf{V}(z) \equiv (B, \tilde{A}, \alpha, \alpha', \chi, \chi')(z) \quad (\text{B.2})$$

at some z_{\max} close to the horizon. The output is some vector $\mathbf{V}(z_{\max})|_{\text{series}}$ which provides initial conditions for the equations of motion given in eq. (A.4). Integrate this system from z_{\max} down to a point z_* in the middle of the bulk. Evaluating the fields and their derivatives at this point produces $\mathbf{V}(z_*)|_{\text{hor} \rightarrow \text{bulk}}$. Now repeat this process from the other direction. Construct the near-boundary series solution using eq. (A.9) and use it to evaluate $\mathbf{V}(z_{\min})|_{\text{series}}$, where z_{\min} is close to the boundary. This provides the initial conditions needed to integrate the system from z_{\min} up to z_* . Evaluating the solution at the last point gives $\mathbf{V}(z_*)|_{\text{bdy} \rightarrow \text{bulk}}$. Finally, take the difference

$$\mathbf{M} \equiv \mathbf{V}(z_*)|_{\text{bdy} \rightarrow \text{bulk}} - \mathbf{V}(z_*)|_{\text{hor} \rightarrow \text{bulk}}. \quad (\text{B.3})$$

derivative. With the fourth order approximation,

$$\frac{F(\chi_{0,0}-2h) - 8F(\chi_{0,0}-h) + 8F(\chi_{0,0}+h) - F(\chi_{0,0}+2h)}{12h} = F'(\chi_{0,0}) + O(h^4).$$

we found that the deviation from the first law of thermodynamics was, at worst, of order 10^{-9} for $\chi_{0,0} \gtrsim 3.7$ and even smaller for lower values of $\chi_{0,0}$.

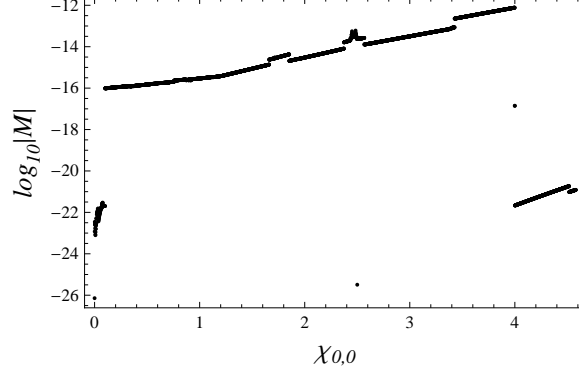


Figure 13. Manhattan norm of mismatch vector, plotted as a function of $\chi_{0,0}$.

This is called the ‘mismatch vector’ [54].

The system of ODEs requires only the values of B and \tilde{A} , and the values and slopes of α and χ , at a single point to completely fix the behavior of the metric and scalar fields throughout the spacetime. If the system is reexpressed as a set of six 1st order ODEs, then a nonzero mismatch vector means that one or more fields are discontinuous across z_* . This implies that our initial choice of \mathbf{X} produced inconsistent initial conditions at the horizon and boundary. The correct choice of \mathbf{X} must lead to $\mathbf{M} = 0$. By thinking of $\mathbf{M}(\mathbf{X})$ as a vector-valued function, the problem becomes that of root finding in six dimensions. We apply the Newton-Raphson method (see, e.g., ref. [55]). It works by a generalization of the familiar one-dimensional method of tracking tangent lines. For a guess \mathbf{X} , compute the Jacobian \mathbf{J} of partial derivatives of the mismatch vector ($J_{ij} \equiv \partial_j M_i$). Then form the new guess $\mathbf{X}_{\text{new}} = \mathbf{X} - \mathbf{J}^{-1}\mathbf{M}$. Iterate to produce a sequence of \mathbf{X} ’s that converge to the true root.

We implemented this algorithm in *Mathematica 7.0*. We let $z_{\min} \equiv d$ and $z_{\max} \equiv 1 - d$ with $d = 1/1000$. We perform a series expansion of the solution close to the boundary, and close to the horizon. Near the horizon the series is evaluated up to and including terms of order $(z-1)^6$ for each of the four supergravity fields. Near the boundary the series is evaluated up to and including terms of order $z^{10} \ln^4 z$ for B , $z^{10} \ln^6 z$ for \tilde{A} , $z^{10} \ln^6 z$ for α , and $z^9 \ln^5 z$ for χ . The equations of motion were integrated using *NDSolve* with *WorkingPrecision* set to 40 digits, *MaxSteps* set to ∞ , and *PrecisionGoal* and *AccuracyGoal* each set to 20 digits. The matching was performed at $z_* = 1/2$. For Newton’s method we found that a step size of $1/1000000$ was adequate to compute a forward finite difference approximation to the Jacobian. We iterated until the Manhattan norm of the mismatch vector, $|\mathbf{M}| \equiv \sum_{i=1}^6 |M_i|$, was below a threshold of 10^{-7} . In practice, for a sufficiently good starting guess for \mathbf{X} , only one or two Newton steps were needed to obtain an incredibly small norm. The data in figure 13 shows how low our norms became after iterating Newton’s method more than a couple times.

For good numerical performance, it is important to independently integrate *inward* from both the boundary and horizon, since $z = 0$ and 1 are regular singular points of the ODEs. To see this, observe that the field equations for α and χ reduce to decoupled Euler differential

equations.²⁹ In all instances, the roots of the indicial equation differ by an integer, and this means that one of the linearly independent solutions has a logarithmic singularity. Any singular behavior is unwanted near the horizon where the fields should be completely regular. Therefore, we seek a solution in which the coefficient of $\ln(z-1)$ vanishes as we approach $z = 1$ from below. However, integrating toward this endpoint is a bad idea since the logarithm grows, and the solution we do want stays constant. Numerical error has the effect of putting noise into the coefficients of the linearly independent solutions, so if the unphysical solution grows faster than the good solution, the relative size of numerical error becomes worse the closer one approaches to the horizon. It is also unhelpful to integrate toward $z = 0$ from above since both linearly independent solutions vanish and a numerically ill-behaved limiting procedure would be needed to isolate the unphysical solution. These practical issues are alleviated by starting from the correct solutions at the horizon and boundary, and integrating into the bulk where no singular points exist. Although our choice for the “middle of the bulk” was arbitrary, we verified insensitivity to its choice a posteriori.

A crucial question that must be addressed in any numerical computation is: how many significant digits in the final result can be trusted? We made extensive use of *Mathematica*’s ability to represent arbitrary precision numbers. Testing showed that integrating with 40 digit working precision was more than sufficient to guarantee that the mismatch vector’s individual components did not change to within 10^{-7} . In particular, our results are also sensitive to the value of d and the truncation order n of the series. The sense in which $d = 1/1000$ is small must be examined in light of how large n is. There is a simple relation between the two: making d smaller is akin to making n bigger since each additional order in a series is roughly suppressed by a factor of d compared to the previous order. We kept $d = 1/1000$ fixed and decreased n to the next nontrivial order for both series solutions. Solving for the roots as before (stopping when a threshold of 10^{-7} for the norm of the mismatch vector was crossed), we found that the new roots were always identical to the original roots in the first 7 significant digits.

To generate good initial guesses for the roots, we linearly extrapolated the last two known roots along the $\chi_{0,0}$ -axis. We found 4581 roots spaced at intervals of 0.001 from $\chi_{0,0} = 0$ to 4.58. For the case $\chi_{0,0} = 0$, the exact root is $\mathbf{X}(0) = (-2, 0, 0, \ln \sqrt{2}, 2, 0, 0)$, which follows from the AdS-Schwarzschild solution.³⁰ Naive guess-and-check was employed to find the first few roots for $\chi_{0,0} > 0$. Although Newton’s method can diverge rapidly if a poor starting guess is made, in practice, we found it to be quite forgiving. Slightly fancier methods involving backtracking were used (quite infrequently) to search for roots when the linear extrapolation

²⁹Near $z = 0$ they become $\alpha'' - (3/z)\alpha' + (4/z^2)\alpha = 0$ and $\chi'' - (3/z)\chi' + (3/z^2)\chi = 0$. Near $z = 1$ they become $\alpha'' + \alpha'/(z-1) = 0$ and $\chi'' + \chi'/(z-1) = 0$.

³⁰A standard form of the AdS-Schwarzschild metric (in units where $L = 1$) is $ds^2 = \rho^2[-f(\rho)dt^2 + d\mathbf{x}^2] + \rho^{-2}d\rho^2/f(\rho)$, where $f(\rho) = 1 - (\rho_h/\rho)^4$ and $\rho_h \equiv \pi T$. This can be rewritten in the form of eq. (A.1) by defining $\rho = \rho_h \sqrt{\cosh 2r}$. The horizon is located at $r = 0$. A further change of variables to $z = e^{-r}$ puts the metric into the form of eq. (A.3), from which we find that $B(z) = (1 - z^4)/(1 + z^4)$. A Taylor expansion around the origin gives $B(z) = 1 - 2z^4 + O(z^8)$ from which we read off $B_{4,0} = -2$. Also, $B_h = -B'(1) = 2$. From eq. (A.11) we find that $A_h = \ln \sqrt{2}$. Lastly, the scalar fields are identically zero in this background.

failed to produce a reasonable guess [55].

C Fluctuation analysis

Tensor channel

For numerical integration it is convenient to consider the system

$$\psi' + (4\tilde{A}' + B'/B - 3/z)\psi + \kappa^2 e^{-2\tilde{A}}\phi = 0, \quad \phi' = \psi. \quad (\text{C.1})$$

Since $z = 0, 1$ are singular points, we use series solutions to provide initial conditions at z_{\min} and z_{\max} , then integrate inward to a matching point z_* . The regular near-horizon solution may be written in the form

$$\phi(z) = c_0 \left[1 + \sum_{n \geq 1} c_n (z-1)^n \right]. \quad (\text{C.2})$$

We evaluate the coefficients up to c_6 . The normalizable near-boundary solution has the form

$$\phi(z) = b_{0,0} z^4 \left[1 + b_{0,1} \ln z + \sum_{n \geq 1} z^n \sum_{k=0}^n b_{n,k} (\ln z)^k \right], \quad (\text{C.3})$$

where we compute the coefficients up to $b_{8,4}$.

Our algorithm for solving the small fluctuation equations is similar to that used for the background and discussed in Appendix B. For a given $\chi_{0,0}$, the near-horizon amplitude c_0 is arbitrarily set to 1 and a guess is made for the eigenvalue κ and the near-boundary amplitude $b_{0,0}$. Next ϕ and its derivative ψ are evaluated with the series at z_{\min} and z_{\max} , which are then fed as initial conditions into `NDSolve`. Using interpolations of the numerically-generated background fields, the system of ODEs is integrated inward to the matching point z_* . We used machine precision settings for `NDSolve`. The mismatch vector for $(\phi(z_*), \psi(z_*))$ is computed and Newton's method in two dimensions is iterated (the Jacobian was calculated with a step size of $1/1000$). Using 5 Newton steps, we obtained Manhattan norms typically several orders of magnitude below 10^{-7} . This yields an estimate for the root κ in units of $1/L$. To find κ in units of πT , we multiply the root by $2/(e^{A_h} B_h)$.

Vector channel

The fluctuation equations in the vector channel are

$$\psi' + (4\tilde{A}' - B'/B - 3/z)\psi + \kappa^2 e^{-2\tilde{A}}\phi = 0, \quad \phi' = \psi. \quad (\text{C.4})$$

The regular near-horizon series solution is

$$\phi(z) = c_0 (z-1)^2 \left[1 + \sum_{n \geq 1} c_n (z-1)^n \right], \quad (\text{C.5})$$

for which we compute the coefficients up to c_5 . The normalizable near-boundary series solution is

$$\phi(z) = b_{0,0} z^4 \left[1 + b_{0,1} \ln z + \sum_{n \geq 1} z^n \sum_{k=0}^n b_{n,k} (\ln z)^k \right], \quad (\text{C.6})$$

for which we compute the coefficients up to eighth order. The numerical algorithm is identical to that for the tensor channel.

Scalar channel

Derivation of gauge-invariant equations for helicity zero

We outline the necessary steps to obtain eqs. (4.15). The $3z$, 00 , 33 , zz , and sum of the 11 and 22 Einstein equations give (in order)

$$H'_{00} + H'_+ + b' H_{00} = -8(3\alpha'_{\text{cl}} \tilde{\alpha} + \chi'_{\text{cl}} \tilde{\chi}), \quad (\text{C.7a})$$

$$\begin{aligned} H''_{00} + (5A' + 2b' + 1/z)H'_{00} + (A' + b')(H'_+ + H'_{33}) + \kappa^2 \frac{e^{-2A}}{z^2} H_{00} \\ = -\frac{8(V_\alpha \tilde{\alpha} + V_\chi \tilde{\chi})}{3z^2}, \end{aligned} \quad (\text{C.7b})$$

$$\begin{aligned} H''_{33} + (5A' + b' + 1/z)H'_{33} + A'(H'_{00} + H'_+) + \kappa^2 \frac{e^{-2A}}{z^2} (H_{00} + H_+) \\ = -\frac{8(V_\alpha \tilde{\alpha} + V_\chi \tilde{\chi})}{3z^2}, \end{aligned} \quad (\text{C.7c})$$

$$\begin{aligned} H''_{00} + H''_+ + H''_{33} + (2A' + 2b' + 1/z)H'_{00} + (2A' + 1/z)(H'_+ + H'_{33}) \\ = -16(3\alpha'_{\text{cl}} \tilde{\alpha}' + \chi'_{\text{cl}} \tilde{\chi}') - \frac{8(V_\alpha \tilde{\alpha} + V_\chi \tilde{\chi})}{3z^2}, \end{aligned} \quad (\text{C.7d})$$

$$H''_+ + (6A' + b' + 1/z)H'_+ + 2A'(H'_{00} + H'_{33}) + \kappa^2 \frac{e^{-2A}}{z^2} H_+ = -\frac{16(V_\alpha \tilde{\alpha} + V_\chi \tilde{\chi})}{3z^2}. \quad (\text{C.7e})$$

The scalar field equations are

$$\tilde{\alpha}'' + (4A' + b' + 1/z)\tilde{\alpha}' + \frac{1}{2}\alpha'_{\text{cl}}(H'_{00} + H'_+ + H'_{33}) + \kappa^2 \frac{e^{-2A}}{z^2} \tilde{\alpha} = \frac{V_{\alpha\alpha}\tilde{\alpha} + V_{\alpha\chi}\tilde{\chi}}{6z^2}, \quad (\text{C.8a})$$

$$\tilde{\chi}'' + (4A' + b' + 1/z)\tilde{\chi}' + \frac{1}{2}\chi'_{\text{cl}}(H'_{00} + H'_+ + H'_{33}) + \kappa^2 \frac{e^{-2A}}{z^2} \tilde{\chi} = \frac{V_{\chi\chi}\tilde{\chi} + V_{\alpha\chi}\tilde{\alpha}}{2z^2}. \quad (\text{C.8b})$$

Hence, there are seven coupled ODEs in axial gauge.

In simplifying the above system it is helpful to use the background equations. These relations make it possible to express second (or higher order) derivatives of the background fields in terms of their first derivatives. The first step is to recognize that H_{33} does not appear in any of the ODEs, only its derivatives, so one more equation is first order in H'_{33} and can be considered a constraint. This allows one to drop one of the 2nd order equations, say eq. (C.7d). The second step is to eliminate H'_{33} from the remaining equations. To do this, sum eqs. (C.7b) and (C.7e), then solve for H'_{33} . As the other independent equation, keep eq. (C.7b). Now plug in for H'_{33} in all four equations. The third step is to use the residual

gauge invariance that preserves the axial gauge, $h_{\mu z} = 0$, to eliminate H_+ . In axial gauge, eq. (4.4), there is a residual gauge invariance parameterized by a vector field $\eta_\mu \equiv \eta_\mu(z)e^{-\kappa x^3}$ which obeys $-\nabla_\mu \eta_z - \nabla_z \eta_\mu = 0$. As a contravariant vector,

$$\eta^\mu(z) = (C_0, C_1, C_2, C_3 + C_z \kappa \int^z dz' e^{-2A(z')}/z', C_z z), \quad (\text{C.9})$$

for arbitrary constants C_μ .³¹ While this preserves $h_{\mu z} = 0$, consider what happens to the other fluctuation components. In particular,

$$\delta h_{00}(z) = -2C_z z B^2 e^{2A}(A' + B'/B), \quad (\text{C.10a})$$

$$\delta h_+(z) = -4C_z z A' e^{2A}, \quad (\text{C.10b})$$

$$\delta h_{33}(z) = 2e^{2A} \left[C_3 \kappa + C_z \kappa^2 \int^z dz' e^{-2A(z')}/z' - C_z z A' \right], \quad (\text{C.10c})$$

$$\delta \tilde{\alpha}(z) = -C_z z \alpha'_{\text{cl}}, \quad (\text{C.10d})$$

$$\delta \tilde{\chi}(z) = -C_z z \chi'_{\text{cl}}. \quad (\text{C.10e})$$

One can easily verify that the fields defined in eq. (4.14) are gauge invariant. Finally, some algebra and judicious rearrangement are needed to obtain eqs. (4.15).

Near-horizon solution

The regular near-horizon series solutions are

$$\tilde{Z}(z) = \tilde{Z}_0 + \sum_{n \geq 1} \tilde{Z}_n (z-1)^n, \quad (\text{C.11a})$$

$$\tilde{a}(z) = \tilde{a}_0 + \sum_{n \geq 1} \tilde{a}_n (z-1)^n, \quad (\text{C.11b})$$

$$\tilde{c}(z) = \tilde{c}_0 + \sum_{n \geq 1} \tilde{c}_n (z-1)^n. \quad (\text{C.11c})$$

It is straightforward to determine the coefficients by solving eq. (4.15) order-by-order in $z-1$. We did this up to fourth order. We find that $\tilde{Z}_1 = -2\tilde{Z}_0$, $\tilde{a}_1 = -2\tilde{a}_0$, and $\tilde{c}_1 = -2\tilde{c}_0$. Higher coefficients are lengthy and we do not write them down.

Near-boundary solution

The normalizable near-boundary series solutions are

$$\tilde{Z}(z) = z \left[\tilde{Z}_{0,0} + \tilde{Z}_{0,1} \ln z + \sum_{n \geq 1} z^n \sum_{k=0}^n \tilde{Z}_{n,k} (\ln z)^k \right], \quad (\text{C.12a})$$

$$\tilde{a}(z) = \tilde{a}_{0,0} + \tilde{a}_{0,1} \ln z + \sum_{n \geq 1} z^n \sum_{k=0}^n \tilde{a}_{n,k} (\ln z)^k, \quad (\text{C.12b})$$

$$\tilde{c}(z) = z \left[\tilde{c}_{0,0} + \tilde{c}_{0,1} \ln z + \sum_{n \geq 1} z^n \sum_{k=0}^n \tilde{c}_{n,k} (\ln z)^k \right], \quad (\text{C.12c})$$

³¹If $C_z = 0$, then the residual gauge freedom simply corresponds to translations in the 4d space (τ, \mathbf{x}) .

where we compute the coefficients up to fifth order. Odd coefficients up to and including $\tilde{a}_{5,k}$ and $\tilde{c}_{5,k}$ are zero. Note that $\tilde{a}_{0,1}$ is not fixed by eq. (4.15) — it is the leading coefficient of the non-normalizable solution. We set it to zero by hand.

Numerical procedure

The shooting method for the coupled set of ODEs cannot start from $z = 0$ or 1 , as these are singular points. Therefore, it is necessary to evaluate the series solutions on the basis vectors at $z = z_{\min}$ and z_{\max} to start the numerical integration. We used settings for NDSolve and Newton’s method identical to those for the tensor channel. However, in this case, Newton’s method only needs to be applied in one dimension, making the calculation much simpler.

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